Working With Patterns Through Chess-Based Problems. Strategies and Reasoning Levels of Primary School Students.

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Abstract

The study of patterns has been recognised for many years as setting up the very essence of mathematics. Patterns are connected to all topics in mathematics, so this theme is present throughout the school mathematics curriculum. Among the large number of interesting examples for working on pattern search in elementary school using situations familiar to students, we chose chess because of the relationships shown between this game and different aspects of mathematics. The objectives were to determine the strategies and classify the students' levels of reasoning when working with patterns to solve chess-based problems. A sequence of activities was designed to carry out this task. The sequence presents visual and numerical patterns ordered progressively from a greater presence of visual aspects to a predominance of numerical aspects. The results of this work suggest that chess favours the use of a variety of strategies, some of them even different from those found in previous literature. Students rely on the geometry of the board when working with these particular types of patterns. However, the results show that the level of reasoning is higher in the case of solving numerical patterns.

Introduction

It is common to find primary schools where chess is played regularly, which facilitates its use to solve problems in mathematics. Several studies support the importance of this game as an educational resource (Ferreira & Palhares, 2008; Sala & Gobet, 2016; Rosholm, Mikkelsen, & Gumede, 2017). In particular, we can introduce various mathematical contents through chess such as combinatorics (Arnal-Bailera & Vera, 2021), arithmetics (Gairín & Fernández, 2010) or patterns (Maz-Machado & Jiménez-Fanjul, 2012).

Patterns are highly present in our daily lives, whether in the decoration of a house or the print on a T-shirt, and we can even find them in nature, for example in the phases of the moon or in the fur of some animals. By pattern, we mean a sequence of objects standing in various relationships (Resnik, 1981), and each object or term will be formed by different elements. Vale (2009) emphasises that mathematical knowledge can be developed through the study of pattern problems and algebra can emerge to generalise and represent this knowledge. Patterns are connected to a large number of mathematical topics which allows their use to develop concepts related to algebra, geometry, probability, among others (Arcavi, 2006). These relationships allow this topic to permeate the entire school mathematics curriculum, both to prepare students for further learning and to develop problem solving and communication skills (Vale, 2009). In particular, there are previous studies about the levels of reasoning of primary school students (Zapatera, 2018), strategies when solving pattern problems (Merino, 2013), and the presence of reasoning and strategies in the case of visual patterns (Barbosa & Vale, 2015).

Working with patterns and educational chess have separately been shown to be of great interest in the study of mathematics teaching and learning. However, not much research has been found analysing the potentials of working on patterns via chess-based problems. Therefore, the aim of this paper is to design a sequence of chess-based problems involving pattern search to analyse how elementary school students reason and what strategies they use to solve these problems. In order to answer this question, we establish the following objectives:

- To determine which strategies appear when primary school students solve visual and numerical pattern chess-based activities.
- To classify the different levels of students' reasoning when they solve visual and numerical pattern chess-based activities.

Theoretical Framework

Patterns: levels of reasoning and strategies

Mathematics curricula around the world attach great importance to working with patterns due to their connections to other topics (Vale, 2009). The National Council of Teachers of Mathematics (NCTM, 2000) establishes that the search for patterns is one of the main learning outcomes in Primary Education, an idea that is also supported by many curricula including the Spanish and Australian, among others. A review of Australian curriculum and textbooks shows a major focus on working with patterns in the early years to carry out activities such as studying visual patterns composed of different shapes and colours (Warren, 2005). Reid (2002) explains that in his work with fifth graders, students examined patterns by observing regularities and they understand that statements must be supported, although they do not yet use mathematically correct reasoning. In this sense, students are more likely to be engaged when working in an environment where certain structures tend to recur frequently. The author assumes that learners expect to find regularities in mathematics that help them to develop and progress in reasoning, and permeate the mathematical activity of the students.

Patterns can be grouped into visual, numerical, linear and logical (Morales, Cañadas, & Castro, 2017). In this study, we focus on the first two. Visual patterns are those in which a geometric regularity can be visually appreciated, such as the movements of the chess knight (see Figure 1). On the other hand, in the case of numerical patterns, relationships between the numbers of the sequence are perceived, such as the number of squares that a bishop threatens according to its position on the board (see Figure 4). This classification is not exclusive, since the same pattern may belong to two types at the same time. In these cases, students have to perceive a regularity based on two different structures, one visual and the other numerical. Radford (2010) explains that the visual structure results from the position of the elements in each object, while the numerical structure results from the number of elements, or other characteristics of the elements described by numbers, such as their position. For example, in activity 2 (see Figure 2), the visual structure would be formed by the squares to which the king can move, while the numerical structure would be the number of these squares. It is clear that understanding the visual structure facilitates the understanding of the numerical one (Healy & Hoyles, 1999) and it is necessary to develop problem-solving skills (Thornton, 2001). In relation to this, when a student is asked to draw a new term of a pattern, Rivera (2010) argues that those who do not coordinate the visual structure with the numerical structure will place the elements of each term in the pattern sequence (squares in the case of activity 2) appropriately, but will not draw the correct number of elements. On the other hand, those who understand the numerical structure but not the visual one, will get the number of elements on each term in the pattern, but will not place them following the visual structure of the pattern.

In working with patterns, it is customary to speak of generalisation, understood by Harel and Tall (1991) as a process that includes the expansion of the applicability range of an existing schema. There are two different types of generalisation requiring different levels of reasoning (Zapatera, 2018): near (to search for nearby terms that can be found through counting, drawings or tables) and far (when it is necessary to find the general rule). An important aspect of generalisation is flexibility in strategy use, i.e. the ability to replace one strategy with another when a task demands change (Nilsson & Juter, 2011). There is some research that supports the treatment of Algebra based on pattern generalisation (Molina, 2009; Radford, 2010) from Primary Education. There are two trends that endorse this idea: (1) on the one hand, pre-algebra serves as a bridge between arithmetic and algebra towards the end of primary school; (2) on the other hand, early algebra aims to work on algebra from the first years of primary school (Zapatera, 2018). Vale and Cabrita (2008) stated that numerical patterns are simpler than visual ones, because in the case of the latter, it is more complicated to obtain an algebraic formula that allows finding a term of the sequence.

Zapatera (2018) established a gradation of pattern generalisation learning consisting of ten cumulative levels. These levels have served to show that 4th grade students are able to express the general rule verbally better than 3rd grade students, although very few 4th grade students are able to express the general rule algebraically (Ramírez, Brizuela, & Ayala-Altamirano, 2020). In a study by Cañadas, Castro, & Castro, (2008), it was found that most students who expressed generalisation, did so verbally. The predominance of verbal generalisation over algebraic generalisation suggests that the former is easier for students. There is evidence that the level of reasoning in near-generalisation tasks is higher than in far-generalisation tasks, which supports the idea that the former are easier for students (Jurdak & El Mouhayar, 2014).

Some of the strategies used by Primary School students in a generalisation task are counting, when the numbers are small, and numerical operations, when the level of complexity increases (Merino, Cañadas, & Molina, 2013). According to these authors, students tend to have more difficulties when using numerical representations, while graphs generally lead to correct answers. Barbosa and Vale (2015) explain that in order to discover near terms in a pattern sequence, students mainly use counting and recursive strategies, while in far generalisations they make greater use of the explicit strategy (discovering a numerical rule, related to the problem context or not, that allows the immediate calculation of any output value given the correspondent input value).

Chess in mathematics education

The game, considered as an educational resource, is closely related to mathematics in general and to reasoning in particular. Therefore, games are considered to support simple deductive reasoning, when deductions go from known premises to a conclusion, and hypothetical deductive reasoning, when deductions go from a premise that is hypothesized to be true, to a conclusion (Reid, 2002). A meta-analysis including 24 studies, showed that chess seems to enhance primary and middle school students' achievement in mathematics and overall cognitive ability (Sala & Gobet, 2016). According to Rosholm, Mikkelsen, and Gumede (2017), replacing the traditional mathematics lesson with a chess-based lesson tended to improve pupils' results in mathematics test scores. The introduction of playful resources such as chess in the mathematics classroom can orient the lessons towards students' interests and increase group motivation (Gairín & Fernández, 2010). Chess favours the understanding of abstract mathematical aspects, chess players must choose the most appropriate strategy at all times and make use of generalisation to act in certain situations (Sala, Gorini, & Pravettoni, 2015). Moreover, chess has been shown to be useful for introducing the idea of pattern in primary education. Maz-Machado and Jiménez-Fanjul (2012) introduced the use of chess in pattern problems similar to our second activity (see Figure 2). Ferreira and Palhares (2008) conducted a study involving problem solving and patterns with primary school students, concluding that students who play chess appear to be the ones who better identify patterns. Additionally, they observed that chess players were better at solving numerical patterns than geometric ones.

Method

Based on the objectives set out in this study, we have chosen to use a qualitative research methodology, a descriptive and exploratory study.

Participants
The study was carried out in a public school in Zaragoza (Spain), where students play chess throughout their schooling, so they already had previous knowledge of this game when the intervention proposal began. Moreover, once a week, one of the Mathematics lessons included chess contents. The area in which the school is located is zoned as middle class. There were 46 students in the 4th year of Primary Education (10 years old) at this school and all of them took part in this study. They were distributed across two classes and the ratio between boys and girls was balanced.

Instrument
A sequence of chess activities has been designed in which different types of patterns are worked on. Due to space limits, a portion of this intervention is presented below (see the Supplementary Materials document for the complete sequence). All the patterns we work with are presented on a chessboard, which gives them a certain visual component. On the other hand, most of them require the use of numbers in some way. For this reason, we have chosen to order the activities from those with a maximum presence of visual patterns to those with a minimum. In all the activities, students are asked to explain the process followed or the reasoning employed, by writing it below the solution. It should be noted that there are no far-generalisation activities due to the young age of the students for whom they were designed.

In the first activity (see Figure 1–left), a board with a knight is shown, this piece can move to the squares of a different colour from the one it is on, jumping over the squares around it (for instance, the knight in Figure 1 can move to the squares with a red circle). The student has to find the rest of the squares it can reach by making one move (red circles), two moves (blue squares) and three moves (green triangles) from where the knight is standing. After solving the activity, the students were asked how they came to their answers. The solution of this activity (see Figure 1–right) consists of a board in which there are red circles in all the white squares, except those surrounding the piece; blue squares in the dark ones, except the corners; and green triangles in all the white squares. Later, they are asked to find how many moves they have to make to reach the squares that the piece has not yet reached, which are the four corners (they need a fourth move). In this first activity, students are asked only about the visual aspects of the solution. For instance, we did not ask them about the number of squares reached by the knight.

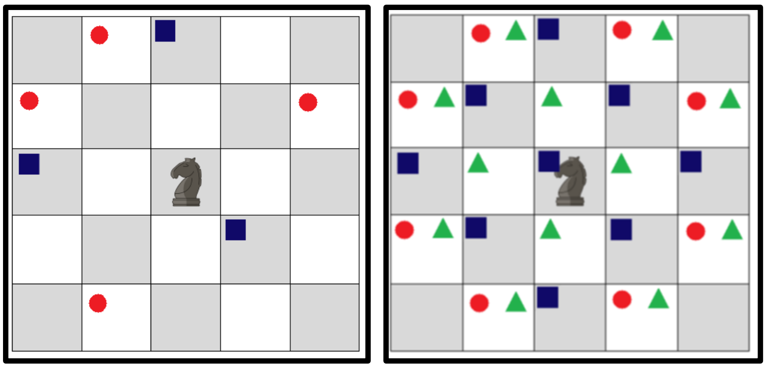

Figure 1. First activity and solution

In the second activity (see Figure 2–left), the student has to find the number of squares the king can reach if it makes three moves while getting as far away as possible from its initial position. The king can move forward one square in any direction. After solving the activity, the students were asked how they deduced the number. The solution of this activity is 24 squares (the 24 squares around the provided chessboard). A board is provided so the students can use it, as in the examples given. In this second activity, there is still an important visual aspect, since the aspect of the solution is a square around the king. However, for the first time, the solution involves the use of numbers.

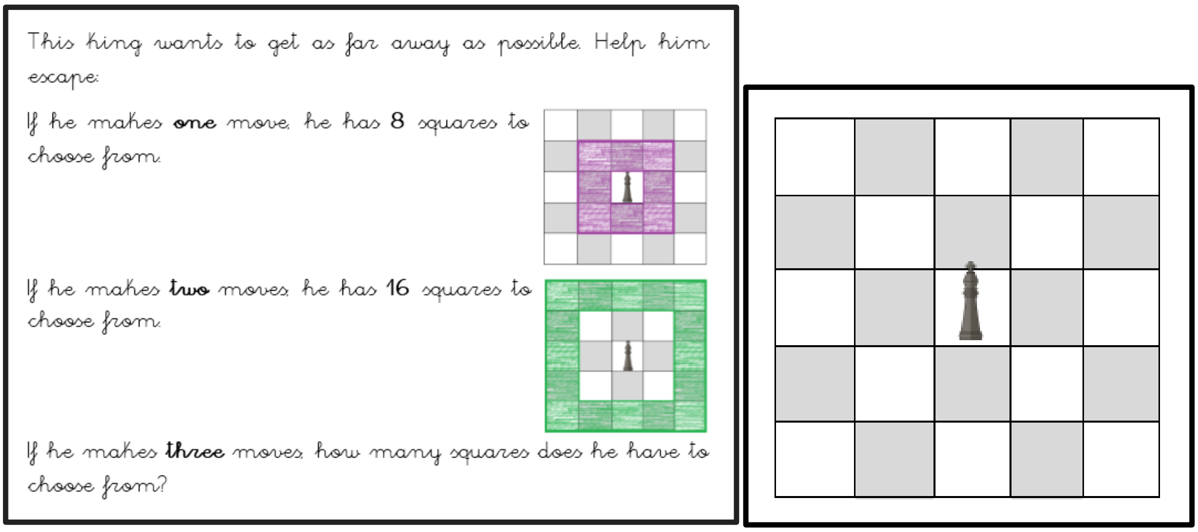

Figure 2. Second activity and board provided

In the third activity (see Figure 3), the student has to find the square where the bishop will arrive, bearing in mind that it always repeats the same sequence of moves. This piece can move any number of squares diagonally. After solving the activity, the students were asked how letters and numbers change. The solution of this activity is square g4. Then, they have to write down the squares through which the piece has passed (for instance, a6, d5, g4) and find the next two squares, in case the board did not end (j3, m2). In this third activity students work with visual aspects, but then have to transform them into an alphanumeric pattern.

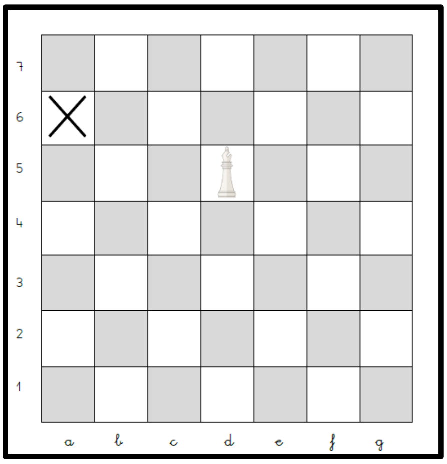

Figure 3. Third activity

In the fourth activity (see Figure 4–left), the student has to complete three 3x3 boards by writing down the number of squares threatened by a rook, a bishop or a queen from each position of the board (a figure threatens a square if this figure could capture another figure in this square with only one move). For instance, the rook threatens 4 squares from the top left corner given that this piece can move forward horizontally or vertically any number of squares, while the queen can move forward in any direction any number of squares. Some numbers were already written in the activity statement as a hint. The solution of this activity is: the rook always threatens 4 squares (those on the same row or column), the bishop threatens 2 or 4 squares depending on the square (those on the same diagonal) and the queen threatens 6 or 8 squares (the queen combines the moves of a rook and a bishop). After solving the activity, the students were asked what relationship they observe between the boards and why they think that happens. In this fourth activity students have to count the number of squares, which prioritises numerical over visual aspects.

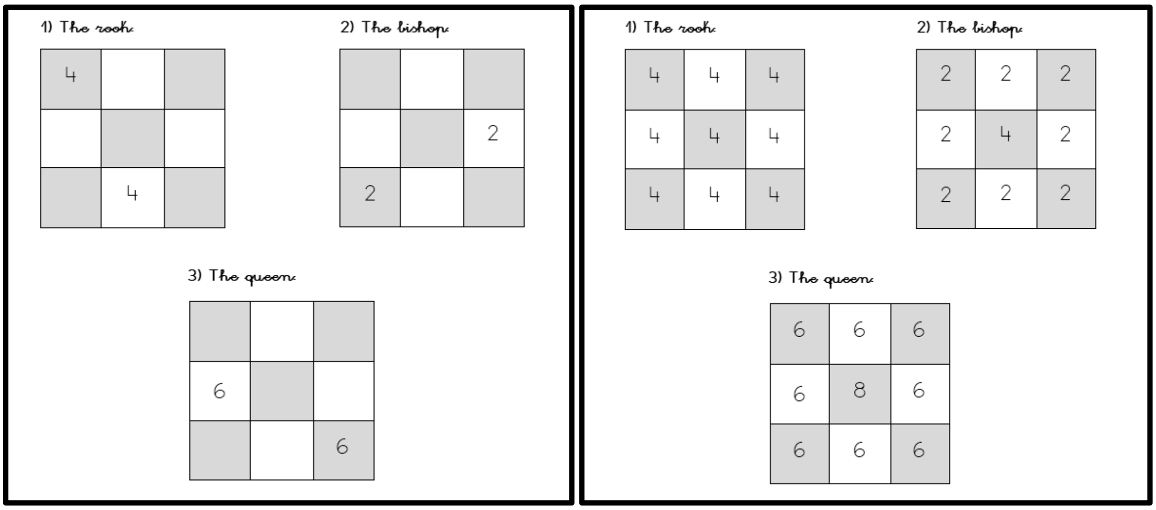

Figure 4. Fourth activity and solution

Variables

The study variables are the strategies used in the search for patterns and the levels of reasoning achieved by the students. In relation to the type of strategies, the categories have been determined in hybrid form (Fereday & Muir-Cochrane, 2006). Some of them have been described in previous literature (Merino et al., 2013; Barbosa & Vale, 2015) while others are closely specific to chess and have emerged during our study. In particular, real game situation and chessboard distribution. They are shown in order from the most elementary to the most elaborate:

- Trial and error: they obtain the solution just by trying out various alternatives until they obtain the one they consider most appropriate, without any further consideration.
- Repetition of the statement: they repeat the instructions given in the statement of the activity.
- Counting: they count the squares through which they have to pass.
- Operation without use of pattern: they use some operation that cannot be related to a pattern for the question.
- Previous step: they only look at the step that precedes it.
- Recursive: they follow the process from the initial situation.
- Real game situation: they proceed as if playing a game of chess.
- Chessboard distribution: they observe how the geometric figures are distributed on the chessboard, and on the basis of this find how the pattern continues.
- Pattern use: they identify a pattern, which can be completely or partially identified and correctly or incorrectly identified.

With respect to the reasoning levels, we have found inductively the following categories:

- Level A. The student does not perform any kind of reasoning.
- Level B. The student performs some kind of reasoning, but they are not able to express it clearly.
- Level C. The student expresses their reasoning, with no reference to arithmetic operations.
- Level D. The student includes arithmetic operations in their reasoning.
- Level E. The student finds the pattern for the case of near terms and reasons how they have achieved it.

Procedure

In order to collect the data, the written productions of the students, in which they have to explain how they have solved the activities, were collected. In addition, the researcher's field diary includes the necessary observations to complement the students' contributions.

A total of four interventions were carried out with each class of students over four different days, with a duration of forty-five minutes each. In each intervention, one of the researchers and the chess teacher were present, the students solved one of the four activities described above individually. The interventions were conducted mainly by the chess teacher, after which, the activities were solved by her, in order to contribute to the learning process.

Data Analysis

In order to extract the categories, both researchers worked separately to analyse the students' responses and classify them according to the strategy used and the level achieved. After this process, they cross checked and discussed their classifications together. The field diary made it possible to review the appreciations noted during the course of the sessions. For instance, it was recorded that some students told the teacher that they relied on their knowledge of chess (chessboard distribution, movements of the pieces, etc.) to solve some of the tasks. This helped to classify the responses of these students.

Results

In the first two activities, the strategies used were analysed, since the students had to explain the process they had carried out to obtain the solution. In the third and fourth activities, the students' level of reasoning was classified, as in this case they were asked to reflect on the solution obtained.

All the students solved the proposed activities, although not all of them were able to explain the questions posed; this could have been due to the young age of the participants and the fact that they were not used to this type of task. As a result, not all of the answers could be classified according to the strategies used or the reasoning produced, broadening the set of strategies proposed in the previous literature. In the first activity, only a visual pattern was presented, with an absence of numerical patterns. The strategies we identified in the students' responses are shown in Table 1.

Table 1. Strategies identified in visual patterns

Strategies	N	Examples
Trial and error	2	S40: "I found out because the knight goes from black to white and I've been trying it out".
Counting	1	S22: "counting the squares, it makes 4 moves, it goes from b1 to a3, from a3 to c2 and from c2 it can go to a1 or e1".
Previous movement	3	S17: "it has to make 4 moves because from some triangle you can go to the corners".
Recursive	5	S19: "following the movements he had done before with the circles, squares and triangles".
Real game situation	2	S20: "I figured it out as if I was playing a game, the moves are from c3 to b1, from b1 to a3, from a3 to c4 and from c4 to e5".
Chessboard distribution	5	S34: "the movement of the circle is in a circular shape, the squares in the shape of a square and on the sides and the triangles in the shape of a rhombus and on top of other figures".

Figure 5 shows how S19 codes the rows and columns of the board with numbers and letters, respectively.

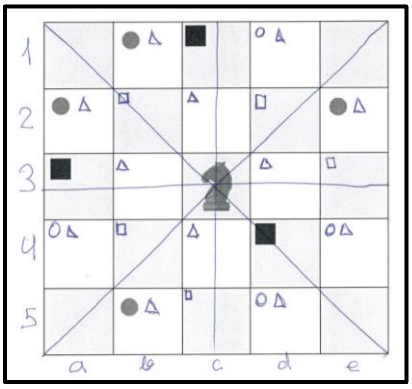

Figure 5. Resolution of activity 1 of S19

The second activity presented a pattern of mainly visual content, but also contained numerical aspects. The strategies used by the students in this activity can be seen in Table 2.

Table 2. Strategies identified in visual patterns with numerical aspects

Strategy	N	Examples
Counting	11	S12: *"I have done it by drawing one more square in each part of the end".*
Operation without use of pattern (additive)	9	S2: "*it has 24 because 16 plus 8 is 24".*
Operation without use of pattern (multiplicative)	6	S40: "*he can go to 24 squares, because 3x8=24 and he has one more fence to jump".*
Use of pattern	1	S15: *"it's 24 squares, because if before it was 8 and 16, it's the board of 8".*

In Figure 6 we see how S12 has added squares around the board and then counted them.

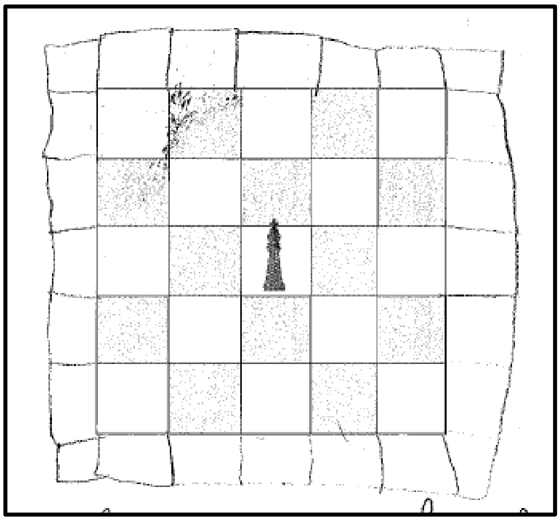

Figure 6. Resolution of activity 2 of S12

It should be noted that several students applied these strategies without using visual support, as in the case of S36 (see Figure 7): *"there are 24 boxes to choose from. I figured it out by adding 16 + 8, which gives 24"*. On the other hand, those who use the counting strategy often need visual support.

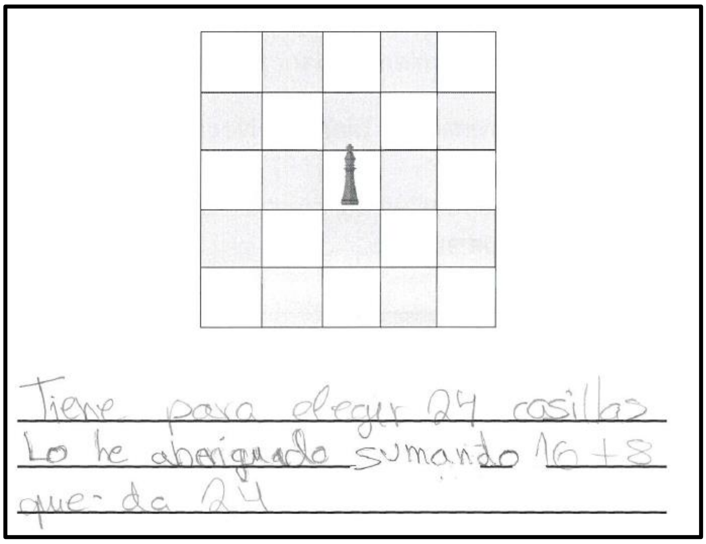

Figure 7. Resolution of activity 2 of S36

In the third activity, a numerical pattern appeared, although with some visual features. Students' reasoning was classified according to the levels shown in Table 3.

Table 3. Levels of reasoning in numerical patterns with visual aspects

Level	Description	N	Examples
A	He/She does not recognise the change in letters and numbers.	6	S4: *"it changes because the board was small and I have made it bigger in my mind"*.
B	He/She recognises that the letters and numbers change throughout the sequence.	14	S19: "*change one number and three letters"*.
C	He/She analyses the situation from the start to find the pattern.	2	S10: "*change in order starting with a letter and a number, but that letter and number has to be smaller than the ones that follow. For example: a, b, c.../1, 2, 3..."*.
D	He/She identifies that numbers change by subtracting one unit.	3	S23: *"numbers are one less and letters are two more"*.
E	He/She finds the correct pattern and is able to reason out the pattern that occurs.	5	S29: *"the numbers change by subtracting 1 and the letters by adding 3"*.

Figure 8 shows the alphanumeric pattern produced by S10.

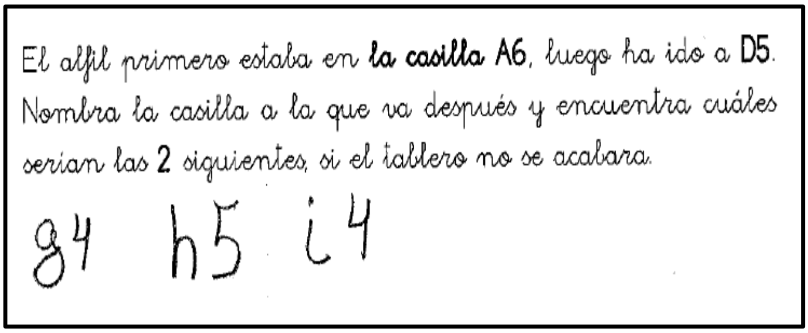

Figure 8. Resolution of the activity 3 of S10

In the fourth activity, we work with a numerical pattern, but it is carried out after having performed a related task in which visual patterns appear. In this case, the levels of reasoning found are detailed in Table 4.

Table 4. Levels of reasoning in numerical patterns

Level	Description	N	Examples
A	He/She is not yet able to express any kind of relationship between the three boards.	6	S14: *"in the centre below are all the boxes full"*.
B	He/She recognises similarities in numbers, but he does not yet explain the relationship.	5	S22: *"many have the same numbers"*.
C	He/She identifies even numbers on the boards in his comments.	3	S18: *"all the numbers are even"*.
D	He/She notes that the numbers on the boards are related by arithmetic operations.	13	S21: "*they add up in pairs"*.
E	He/She expresses that the relationship that links the boards is that the queen's move is the sum of the rook and bishop moves.	7	S19: *"the relationship between them is that rook + bishop = queen"*.

Discussion and conclusions

The objectives of this work were to determine which strategies appear when primary school students solve visual and numerical pattern chess-based activities and to classify the different levels of reasoning when they solve these activities. For this purpose, a sequence of activities was designed including chess-based problems, with different degrees of numerical and visual aspects. This proposal was carried out with pupils in the 4th year of Primary Education who have been practising chess since Infant Education, but who had not previously carried out pattern tasks.

With regard to the first objective, counting has been the most used strategy in working with numerical patterns. This coincides with Merino et al. (2013), who argue that this strategy is commonly used when dealing with small numbers, while when the numbers are larger, students draw on other types of strategies, such as the application of numerical operations. The latter is reflected in this study, as some students used operations such as addition or multiplication instead of counting. Barbosa and Vale (2015) corroborate the priority use of counting, but when it comes to a visual pattern they highlight the recursive strategy, as happened in this work when students attended to previous movements. As far as visual patterns are concerned, in contrast to previous studies, we have found chessboard distribution as one of the most common strategies. We consider that the appearance of chess-related strategies (chessboard distribution and real game situation) is associated with factors (Lannin, Barker, & Townsend, 2006) related to cognition (students' familiarity with chess) and tasks' structure (chess-based problems). One aspect to highlight is that the more complex the strategy used, the less visual support the students needed. The variety of strategies found, some of them different from those found in previous literature, is a sign that the activities designed allow for different ways of solving them, an aspect that favours flexibility (Nilsson & Juter, 2011).

As for the second objective, we have detailed five levels (A-E) related to the ten levels proposed by Zapatera (2018), specifically our level A corresponds to Zapatera's level 0 (the student does not continue the sequence), and our level E corresponds to Zapatera's level 2 (the student performs near generalisation), since this study focuses on near generalisation, Zapatera's level 1 (the student continues the sequence) has been divided into three different levels (B-D) according to the reasoning level of the student. Students tend to use their knowledge of chess to solve the proposed activities, for example some students were able to identify the queen's move as the sum of the rook's and bishop's moves. Other students referred to the chessboard as an instrument which helped them to solve some of the number pattern tasks. The field diary helped to identify this, as some students commented: "the chessboard that appears in the activity is small, but I have made it bigger in my head and so I have been able to continue the move". In addition, not including far generalisations made it easier for students to solve the task, an idea supported by Jurdak and El Mouhayar (2014), who recognise that working with near generalisations is easier for students. On the other hand, Zapatera (2018) recommends introducing far generalisation from 3rd grade of Primary School, because it presupposes previous work with patterns. The students who participated in this research had no prior knowledge of pattern problems, so the fact that they were able to solve the presented activities suggests that chess could be considered a suitable resource to introduce pattern generalisation reasoning at this level.

One of the most interesting results found in this work is the relevance of the relationship between the game of chess and geometry. In our study, students used strategies in which they showed knowledge of various geometrical aspects of the chessboard, for example when describing the distribution of the pieces and the moves made. Ferreira and Palhares (2008) observed that chess players performed better in numerical pattern tasks than in visual ones. However, the students in this study, who have played chess since Infant Education, tend to rely on geometric aspects to solve both visual and numerical pattern problems. This idea of combining the work with different types of patterns, visual and numeric, is supported by authors such as Morales et al. (2017), who recommend introducing logical patterns from an early age, to allow students to recognise the existence of different types of patterns.

When students explained their strategies and solving processes, there were more answers in the case of patterns that include some numerical aspects. Moreover, these answers were better explained, showing higher reasoning levels. Cañadas et al. (2008) identified similar situations, with students preferring to make use of numerical representations, despite the fact that the statement presented contained graphical representations. At the root of this trend could be the preference given throughout primary education to arithmetic operations in comparison to other content in mathematics in which patterns could be worked on, such as Geometry or Algebra (Molina, 2009). Pattern generalisation tasks are common in primary school in some countries such as Australia (Warren, 2005), other curriculum (NCTM, 2000) suggest that pupils in grades

3rd to 5th should be able to express numerical and geometrical patterns in words or symbols, as well as to analyse the structure of the pattern to identify the changes that occur and to recognise how the sequence would continue.

Limitations of the study

It has been shown that, under the conditions of our study, chess-based problems are a suitable context to work with near generalisations, an aspect that can help to improve the Primary Education curriculum in Mathematics. However, this study has not explored the possibility of including far generalisation tasks, so another future line of research would be to analyse the impact of chess-based problems in this type of generalisation. In particular, what type of strategies students would use and if these would be different from those used in contexts other than chess (Barbosa & Vale, 2015). Finally, we have found that students showing higher reasoning levels with numeric patterns are not the same as the students showing higher levels with visual patterns. Therefore, it would be interesting in future studies to explore which characteristics are shared by students who solve numerical or visual patterns better.

Implications for teaching and learning

Huizinga (1988) was the pioneer of game-based learning, based on this concept Tobias, Fletcher, and Wind (2014) explored evidence on the effectiveness of using games to provide instruction, concluding that an overlap must exist in the cognitive processes engaged by both. In this regard, since we want to take advantage of the fact that children like to play chess, game-based learning has to include the designing of learning activities that can incrementally introduce concepts exploiting the potential of the game. In this respect, Sala and Gobet (2016) explain that the skills acquired by the chess player are very context-dependent, although in the case of young children, they are less dependent than in the case of adults, so for example, the transfer of learning should be easier in Primary than in Secondary. Ifenthaler, Eseryel, and Ge (2012) suggested the need for a systematic study of which instructional design strategies work best in game-based (chess-based in our case) learning environments. In particular, we have to start designing far-generalisation activities using chess-based problems to work with students with different levels of knowledge of the chess game. We consider that game-based learning can provide a useful theoretical framework for this.

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