Of course the actual thrust against each abutment is exerted in an oblique direction, and it is from the corresponding oblique reactions, one of which is shown at the right abutment in figure 1, that the stresses in the several members of the truss will be ultimately determined; but, for the present, it is more convenient to consider the effects of the vertical and horizontal reactions into which each of those oblique reactions may be resolved.

Let the vertical reactions of W_2 (fig. 4) be denoted by v and v′, the magnitudes of which may be obtained as in the case of an ordinary girder. The effect of v is supposed to be transmitted through the several members of the truss from the left abutment to the point of application of W_2, and that of v′ is considered as being similarly conveyed from the right abutment.

Let the horizontal components of the reactions developed at the left and right abutments by W_2, be respectively denoted by h and h′. These horizontal reactions are obviously equal to each other in magnitude; but, for the sake of uniformity of notation, as well as for the purpose of promoting perspicuity, it is considered desirable that they should be distinguished by appropriate indices.

Let the letter L denote the length of the span; l, the length of any member; a, the sectional area of any member; s, the stress in any member produced by v or v′; s′, the stress in any member produced by h or h′; and S, the resulting stress in any member, being the sum or difference of s and s′.

If the truss were erected as represented in figure 4, so that its extremities would be free to move horizontally, it is manifest that an alteration in the length of any member, G g for instance, would be accompanied by a corresponding alteration in the length of the span; and, assuming every other member to be absolutely rigid for the time being, it is also manifest that this alteration in the length of the span which we shall call L′ must be proportional to the alteration in the length of G g which we shall desingnate l′. From the existence of this proportionality it follows that the ratio $\frac{L'}{l'}$ must possess a constant numerical value which we shall denote by the letter q. Hence

$$\frac{L'}{l'} = q \qquad (2)$$

or

$$L' = q \times l'$$

but by formula 1

$$l' = \frac{S \times l}{a \times E}$$

therefore
$$L' = q\frac{S \times l}{a \times E} \tag{3}$$
in which S is the stress in Gg.

Again, since either half of the truss may be regarded as a bent lever having its fulcrum at the point of intersection of f g and g i, it necessarily follows that the vertical reaction v must be directly proportional to the stress s which it produces in G g ; and therefore, the ratio $\frac{s}{v}$ must possess a constant numerical value which we shall denote by the letter p. Hence

$$\frac{s}{v} = p$$

or
$$s = v \times p \tag{4}$$

If this value of s be substituted for S in equation 3, we shall obtain the extension of the span due to the compression produced in G g by the reaction v. Thus

$$\text{Elongation of span} = q\,p\frac{v \times l}{a \times E}$$

By proceeding in a precisely similar manner the extension of the span due to the alteration in length of each member produced by the reaction v or v′ may be obtained ; and then, the total extension of the span which would result from the effects of W_2 if the truss were free to move horizontally on its bearings, may be expressed thus :

$$\text{Total extension of span due to } W_2 = \text{sum of } q\,p\frac{v \times l}{a \times E} \tag{5}$$

But the truss is not erected as represented in fig. 4. It exerts a thrust against each of the abutments which are assumed to be perfectly rigid, and its span cannot become increased in length owing to the horizontal reactions of these abutments, the effects of which we shall now proceed to investigate.

The horizontal reaction h, if acting alone, would produce a tensile stress (s′) in the member Gg, and, owing to either half of the truss acting as a kind of lever, pivoting round the point of intersection of fg and gi, it follows that

s′ is proportional to h

and it has been shown that

L′ is proportional to l′

L' being the alteration in length of span due to the alteration (l') of G g, which would be produced by the stress s'

$$\text{therefore } s' : L' :: h : l'$$

$$\text{or} \quad s' : h :: L' : l'$$

$$\text{therefore} \quad \frac{s'}{h} = \frac{L'}{l'}$$

$$\text{but by formula 2} \quad \frac{L'}{l'} = q$$

$$\text{therefore} \quad \frac{s'}{h} = q$$

$$\text{or} \quad s' = q \times h \qquad (6)$$

On substituting this value for S in formula 3, we obtain the contraction of the span due to the elongation which would be produced in G g by the horizontal thrust h, if acting alone. Thus—

$$\text{Contraction of span} = q^2 \frac{h \times l}{a \times E}$$

and since the stress in each member of the truss due to h or h′ produces a corresponding contraction of the span, it follows that the total contraction of span due to h and h′, if acting alone, would be expressed thus—

$$\text{Total contraction of span due to h and h}' = \text{sum of } q^2 \frac{h \times l}{a \times E} \qquad (7)$$

But, owing to the assumed rigidity of the abutments, the span cannot become altered in length; hence, the extension of it which would be produced by v and v′ must be exactly equal to the contraction of it which would be produced by h and h′.

$$\text{Therefore, sum of } qp \frac{v \times l}{a \times E} = \text{sum of } q \frac{h \times l}{a \times E}$$

$$\text{and, denoting} \frac{l}{a \times E} \text{ by } r$$

$$h = \frac{\text{sum of } q\,p\,v\,r}{\text{sum of } q^2\,r} \qquad (8)$$

On Plate 15 the Greek letter sigma denotes "the sum of"; and the reaction of the right abutment is denoted by v′ and not v as given at the bottom of plate.

In order to apply this formula it is essential that the sectional area (a) of each member of the truss should be known, as would be the case when the problem to be solved consisted in investigating the stability of an existing structure, of which it is convenient, for the present, to consider the truss delineated in fig. 1 as a representation. Once the sectional areas of the several portions of the truss are known the value of r for each member may be readily ascertained, and hence, in order to obtain the magnitude of h with the aid of formula 8, it is only necessary that the values of p and q for each member of the frame should have been previously determined.

If the vertical reaction v be assumed equal to unity in formula 4, we would have

$$s = p$$

that is, the value of the constant ratio p for any member of the truss is identical with the stress which would be produced in that member by a unit vertical reaction, and hence we have the following rule for determining the several values of p :—

Determine the magnitude of the stress produced in each member of the truss by a unit vertical reaction, then these magnitudes numerically express the respective values of the constant ratios due to a vertical system of loading.

Again, if h be assumed equal to unity it follows from formula 6 that

$$s' = q$$

and hence the following rule for determining the several values of q :—

Determine the magnitude of the stress produced in each member of the truss by a unit horizontal reaction, then these magnitudes numerically express the respective values of the constant ratios due to a horizontal abutment reaction.

After having ascertained the values of p and q for each member of the truss, by drawing stress diagrams similar to those represented in figures 5 and 6, which have been respectively constructed for a unit vertical and unit horizontal reaction, a table resembling that headed No. 1 should be made out. Of course if the two halves of a structure were dissimilar, such a table would have to include every member of the frame, but if they are similar, as in the example under notice, those members composing either half need only be tabulated.

When filling up the columns under W_1, W_2, &c., care should be taken to use the proper value of v, namely, the reaction of the left abutment when dealing with a member situated on the left of the point of application of the weight at the head of the column, and the reaction of the right abutment when dealing with a member situated to the right of that point. In the case of both of these reactions being common to the same member, their mean should be tabulated.

Table No. 1 illustrates the mode of procedure which should be adopted in order to obtain the magnitude of the abutment reaction due to one or any number of the weights W^1 W^2 &c., and by compounding this horizontal reaction with the total vertical reaction, the magnitude and direction of the required oblique abutment reaction may be readily determined. Then from this oblique reaction the actual stresses produced in each member of the truss may be derived by constructing a single stress diagram similar to that delineated in Fig. 7. They may also be arrived at by taking the algebraic sum of the stresses which would be induced by the vertical and horizontal reactions considered as acting independently, in which case two stress diagrams, somewhat similar to those shown in Figs. 5 and 6, would have to be constructed.

Table No. 2 shows the theoretical saving of material which would probably be realised by making the truss abutting in place of permitting it to move freely on its bearings.

A more typical form of abutting truss than that delineated in Fig. 1, might easily have been chosen for the purpose of illustrating the method of calculating abutment thrusts set forth in this paper. But the example, to the consideration of which your attention has just been directed, has already been investigated in a most minute and elaborate manner by Professor R. H. Smith; and it is solely on this account that it has been selected on the present occasion, in order that a comparison might be readily instituted between two widely different modes of procedure, and between the results derived from an application of each of these modes to the solution of one and the same problem. It would, of course, be futile and unreasonable to expect that identical results should follow from these two distinct and dissimilar modes of calculation, because absolute mathematical accuracy is seldom if ever attainable in conducting any of those complex operations which more or less frequently command the thoughtful consideration of those whose avocations lead them to labour in the vast and eminently interesting field of applied mechanics. However, on referring to page 148 of the 49th volume of the *Engineer*, it will be found that the abutment thrust, as calculated by the gentleman whose name has just been cited, is practically equal in magnitude to that deduced from table No. 1; and it will also be found that the sectional areas computed in both instances, if not identical, at least exhibit an approximation to being so which should satisfy the desires of even the most fastidiously disposed engineer.

It rarely happens that a person engaged in designing framed engineering structures finds himself called on to inquire into the stability of an existing truss, the sectional areas of whose members are given quantities. On the contrary, the determination of these same sectional areas almost invariably constitute the problems which present themselves for solution in this important branch of engineering. In abutting trusses, these sectional areas, as already remarked, cannot be calculated either directly or with anything approaching to what would be styled mathematical precision; and in order that they may be determined within a satisfactory degree of approximation by utilising formula 8 or any other similar formula constructed for the purpose of attaining the same end, it is essential that each member of the truss should have been previously assigned some particular sectional area of a probably correct value; then the accuracy of these assumed values may be tested and the proper ones deduced by proceeding in th following manner:—

Either assume or determine approximately a sectional area for each member of the truss. From these sectional areas and the given system of loading, calculate the horizontal thrust as already described; and, for the horizontal thrust thus deduced, determine the proper sectional areas which should be assigned to each member of the frame. If these latter areas practically agree with the assumed values, the problem is solved; but, if not, the assumed areas must be modified, and the operation repeated until a satisfactory degree of equality is obtained; one, or at most two repetitions will generally be found sufficient.

No doubt, the task of designing an abutting truss is, under existing circumstances, both tedious and troublesome; but the author feels confident that the many advantages possessed by this type of structure when erected under favourable conditions, will result in speedily bringing it into favour with engineers. Precedents will thus be established, tables of weights will be compiled, deformations will be observed and recorded, and many other valuable statistics shall be rendered available which will tend to materially simplify and facilitate the prosecution of an undertaking that is now fraught with many delays and inconveniences. Even if this important problem were not capable of simplification, the preeminent adaptability of the arch truss to special localities would unquestionably warrant the incurring of whatever additional trouble the process of designing it may entail.

Approximate Determination of Horizontal Thrust.

The truss represented in fig. 8 has been designed to carry a permanent bridge load of 1 ton per foot run of span, and also a moving load of the same intensity. Assuming that the moving load covers the whole span, each of the weights W W_3, etc., would have a magnitude of 20 tons, while W_1 and W_{12} would each have a magnitude of 10 tons. Then if a corresponding line of weights (fig. 9) were set off, and the polar distance (HO) selected so as to represent the horizontal abutment thrust, it is well known that the corresponding polar polygon passing through the extremities of the truss would constitute what is termed the line of pressure or line of equal horizontal thrust. Consequently, if we knew the exact value of the horizontal thrust developed by the given arrangement of loading, we could at once make a faithful delineation of the true line of pressure; or, *vice versa*, if we were enabled to produce a correct representation of the true line of pressure, we could immediately deduce from it the exact value attained by the actual horizontal thrust. But, so long as the magnitude of this horizontal thrust remains undetermined, it is impossible to trace out the corresponding true line of pressure, except in the case of the braced arch where it passes through three fixed points. However, owing to our knowledge of the fact that it tends to coincide with what Moseley has styled the line of least resistance when treating of the solid arch ring, we can, by drawing a line of pressure so as to pass through a point situated near the central bay of the upper flange, approximately ascertain the value of the horizontal thrust; and, from this approximate magnitude, the sectional areas of the several members of the truss may be deduced with sufficient accuracy to render them applicable to being formed the basis on which the more elaborate and exact method previously described may be founded. To obtain this approximate value of the horizontal thrust, the mode of procedure would be as follows:—Draw a vertical line

(PQ) through the centre of gravity of the weights on one-half of the truss: this vertical is the line of action of the resultant of these weights. Select a point in the centre line of the truss and near the central bay of the upper flange. Through this point draw a horizontal line to intersect PQ; and join this point of intersection (K) with the left extremity (A) of the truss. Set off the line of weights FG (Fig. 9) to represent the load on the whole truss; through the centre point (H) of FG draw a horizontal line HI; and through F draw the line FO parallel to AK. Then, the line HO will represent the required approximate value of the horizontal thrust, and from it the corresponding line of pressure may be drawn as shown by the thick dotted lines in figure 8, and the probable sectional area of each member deduced by constructing a stress diagram similar to that shown in figure 7.

The approximate value of the horizontal thrust, obtained in accordance with the method just described, for the truss delineated in figure 1, is shown by the line ZG in figure 2; and the line of pressure to which it corresponds is shown by the thick dotted lines on the former figure. The true line of pressure is drawn in thick full lines on figure 1; and the correct magnitude of the horizontal thrust, as obtained from table 1, is shown by the line ZG in figure 3. It will be observed that the magnitudes of the horizontal thrust as given by figures 2 and 3, only exhibit a discrepancy to the insignificant amount of 0.5 tons, thus demonstrating the feasibility of obtaining a very close approximation to the correct sectional areas of the members of an abutting truss from a careful application of the method which it is to be hoped has been clearly elucidated in the preceding paragraph. Of course it may happen in some instances that the approximate value of the horizontal thrust, derived from this method, does not agree so closely with its true value; but by exercising a little judgment, it may always be found with sufficient accuracy to enable us to proceed with operations that lead to more correct results, and to construct a line of pressure capable of affording reliable indications as to whether the truss would be stable or not in its existing form.

However, although the maximum horizontal thrust will be developed when the moving load covers the entire span, still the maximum stress in every member would not be that due to this arrangement of loading. On the contrary, some members are much more severely strained when the moving load covers only a portion of the span, so that before the probable sectional areas derived from the former disposition of loading be constituted the basis of operations for the prosecution of any elaborate or accurate calculations, they should be slightly modified in a manner about to be explained. For our purpose it will be sufficient to consider the moving load as covering exactly half the span ; and in this particular case, as well as in every other case of unsymmetrical loading, that portion of the line of pressure passing through the vertical axis of symmetry of the truss will not be horizontal, as in the preceding example ; and, in order to construct a line of pressure so as to pass through some particular point in this axis or centre line of truss, as well as through its two extremities resting on the abutments, *i.e.*, through three fixed points, it is necessary to proceed as follows :—

With any pole O′ (Fig. 11) draw the polar polygon shown in Fig. 10, and produce its extreme sides until they intersect. This point of intersection will be situated on the line of action of the resultant of the total load bearing on the truss (see author's paper on graphic statics.) Extend the side passing through the centre line of truss in both directions until it intersects the extreme links of the polygon produced ; and through these latter points of intersection draw the verticals PQ and RS. Take a convenient point (E) in CD, and join it with the points A and B ; also join the points M and L where the lines AE and BE respectively intersect the verticals PQ and RS ; and produce LM to intersect a production of AB in J. Select a point (T) in the centre line of truss through which the true line of pressure would be likely to pass ; and from J draw a line through T to intersect the verticals PQ and RS in the points U and V respectively. Then, if through F and G (Fig. 11) two lines be drawn respectively parallel to AU and BV, they will intersect in a point O, which is the pole of the required line of pressure passing through A, T, and B. The lines OF and OG represent the magnitudes and directions of the oblique abutment thrusts. whilst the horizontal line to FG represents the magnitude of the corresponding horizontal thrust, from either of which thrusts the approximate values of the several members of the truss may be deduced as already explained. In selecting the point T, it is, of course, necessary to exercise some judgment, which may, however, be acquired from a comparatively brief experience in the delineation of these lines of equal horizontal thrust.

By comparing the sectional areas of the various members obtained from this unsymmetrical arrangement of loading with those previously calculated on the supposition that the moving load was uniformly distributed over the entire span, a sufficiently close approximation to the true values of the sectional areas may be arrived at to enable us to proceed, under favourable conditions, with the construction of a table similar to that headed Table No. 1. And, in passing, it may be well to remark that, in compiling such a table, the lengths of the several members of the truss must be obtained as aacurately as possible. Consequently, unless these quantities are determined by calculation, the truss diagram from which they are derived should be drawn to a scale of not less than eight or ten feet to an inch. Also, the stress diagrams, with the aid of which the values of p and q may be advantageously computed, should not be drawn to a scale of less than one or two inches to a ton, and the values of r should be carefully worked out.

The method of drawing a line of pressure through three fixed points which has just been described, is especially applicable to what is termed the braced arch. In dealing with this form of structure it is not necessary to draw the the line of pressure through an assumed point situated near the centre of the span as in the preceding example, for in this particular case the two springing points together with the centre of the hinge connecting the two portions of the arch at the crown, constitute three permanently fixed points through which the true line of pressure must pass, and from it the exact value of the actual horizontal abutment thrust, or of the oblique abutment thrust, may be readily deduced, as already explained in the preceding paragraphs.

In concluding, the author desires to offer some apology for whatever omissions and imperfections his paper may be found to contain. The subject of which it treats, besides being comprehensive and abstruse, is deeply entangled in the closely woven web of mystery with which it has been surrounded by mathematical experts. It would furnish ample matter for the compilation of a fair sized volume, and, at most, only its salient features could be touched on in a paper attaining even to the prolixity of that which has been introduced to your notice this evening. However, a want has long been experienced for the enunciation of some simple method of calculation which would enable ordinary professional practitioners to utilise the modern form of the "metallic arch," or as it is sometimes styled the "elastic arch." Such a method should be capable of affording, though not necessarily accurate, at least practically correct results. Its application should be independent of the higher branches of mathematics, of which few of those who are busily engaged in the prosecution of a general professional practice can find time for maintaining their knowledge in an exemplary condition of efficiency, and it should be free from any complications that might lead to troublesome or aggravating errors. To give publicity to a mode of determining the proper sectional areas for the several members of an abutting truss that seemingly fulfils these conditions has been the object of the author in preparing the remarks which he has just read for you; and, if his efforts should not prove successful, it is his earnest hope that they may at least pave the way for the labours of more capable and accomplished investigators.

The Paper was illustrated by a number of diagrams, from which plates 14 to 16 have been prepared.