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Linguistic differences and problem-solving routines in mathematics

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The purpose of this study is to investigate routines as guides for mathematical thinking. Four English-speaking and four Korean-speaking students were interviewed in English about the concepts of limit and infinity. Based on the communicational approach to cognition, which views mathematics as a discourse, we identified the primary characteristics of students' routines for infinity and limit. Results show that language differences between English and Korean affect students' problem-solving routines in mathematics. On the basis of these results, we conclude that there is a need to deal with linguistic sensitivity in mathematics learning.

Keywords: discourse analysis; English and Korean language differences; mathematics; problem-solving routines

INTRODUCTION

Language is one of the important tools teachers use to communicate new concepts to students; new concepts are explained in context through the use of spoken and written words and symbols. In the process of such explanations, students develop routines that pattern their thinking; the use of words and symbols at the object-level, and routines at the meta-level correlate in the process of learning (Sfard, 2008). Routines play a role in this interwoven process when students learn mathematics,

A basic assumption in this study is that meta-discursive rules in meta-level activities guide students to think in certain patterns. In this study, routines are regarded as meta-discursive rules that exemplify regularities in students' discourses. There are two important aspects to the definition of discursive routines: when the routine should be used; and how routines should be implemented (Sfard, 2008). *When* refers to the cues for beginning discursive routines. *How* refers to the kinds of patterns which exist in discursive routines. Mathematical discourses are composed of a number of object-level and meta-level activities (Sfard, 2008).

We employed a discourse analysis methodology in our study to examine the role of language in object-level learning and the impact of language differences on meta-level learning. The reason for undertaking a linguistic comparison between English and Korean, is because English embodies a continuity in lexical development between colloquial and mathematical discourses in the uses of mathematical words but Korean does not. Thus, we assumed that the linguistically different mechanisms in lexical development between English and Korean may account for differences in students' object-level learning and mathematical routines in meta-level activities.

THEORETICAL FRAMEWORK

Learning the notions of infinity and limit and their epistemology

In the history of mathematics, the concept of infinity is interwoven with the concept of limit. In spite of their mutual interdependence, there is little research on students' simultaneous understanding of, and difficulties with, these concepts. Various aspects of learning about these concepts, however, have been investigated over the last few decades, with researchers identifying reasons for learning difficulties associated with mathematical structure (Borasi, 1985; Cornu, 1992; Cottrill et al., 1996; Tall, 1992; Tirosh, 1992; Vinner, 1992), misconceptions and cognitive obstacles (Davis & Vinner, 1986; Fischbein, 2001; Fischbein, Tirosh, & Hass, 1979; Przenioslo, 2004; Williams, 2001) and cognitive theory (Tsamir & Dreyfus, 2002; Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). The studies have provided important insights into mathematical learning and teaching but have not led to satisfactory solutions to students' learning problems. Discourse analysis may be a means of furthering understanding of the learning problems, especially, in the case of this study, in the role of language in routines at the meta-level.

Conceptual framework

We considered two main issues when establishing a theoretical framework for this research: the role of language in the consideration of object-level learning, and the impact of linguistic differences on the mechanism of meta-level activities.

The nature of language in object-level learning

In Vygotsky's (1978) opinion, speech for communicating with others comes before internal speech in children's internalization of higher mental processes:

> Every function in the child's cultural development appears twice: first on the social level, and later, on the individual level; first between people (inter-psychological), and then inside the child (intra-psychological). (p. 57)

Thus, because of the inherently social nature of human activities, thinking arises from an individualized version of interpersonal communication; thinking is communication with oneself (Sfard, 2008). Learning is sensitive to contexts, including society, culture, and situations (Nunes, Schliemann, & Carraher, 1993; Rogoff, 1990). Cobb (1994) believes that because of the social nature of student learning, it should be "a process of enculturation into a community of practice," not "a process of active cognitive reorganization". Therefore, learning is the act of becoming a participant.

To be a participant in a community of practice, such as in learning mathematics, mathematical tools, such as symbols and mathematical language, are important and enable shared consciousness (Hersh, 1997); they are a collection of tool-mediated products. As Vygotsky (1986) points out, higher mental processes are not only developed through the procedures of internalization of public speech to inner private speech, but also tightly related to tool-mediated activity. The properties of these tools are inseparable from the cognitive processes related to the uses of the tools (Rogoff, 1990): thought must be transferred through meanings and only then through words as a tool. According to Vygotsky (1986), "the word is a direct expression of the historical nature of human consciousness" (p. 256). In other words, consciousness can be investigated in a word and thus thought and language are inseparable.

Routines in meta-level learning

If the aim of learning mathematics is to become a more skilful participant in mathematical discourse, two important factors deserve particular attention: the mediating tools that students use in their mathematical discourse, and the meta-discursive rules that regulate their mathematical discourse in certain patterns (Sfard, 2008). The mediating tools deal with the object-level activities in mathematical discourse, and the meta-discursive rules guide the meta-level factors of mathematical discourse. When students use mathematical keywords and solve problems, it is possible to detect certain discursive routines.

DESIGN OF STUDY

Research questions

The study was designed to characterize the ways students think about the mathematical concepts of infinity and limit, and has two aims:

- Examine the primary characteristics of routines of native English and Korean students' discourse on infinity and limit.

- Examine the differences between the discourses of two linguistically distinct groups of students on infinity and limit.

Methodology

Study participants were divided into two ethnically distinct groups. Each group included one elementary student, one middle school student, one high school student, and one university undergraduate (groups members were tagged with symbols for reference, such as A10 for the American 10^{th} grader and KU for the Korean undergraduate.). Elementary school students were included not only because we searched for differences in the mathematical discourses of different age groups within and across the ethnic groups, but also because English-speaking students encounter the words infinity and limit in everyday life. The four American students were English speakers from the US, while the four Korean students were English speakers whose first language is Korean. Data were collected on the basis of one-to-one interviews in English using open-ended questions. The interview questionnaire consisted of seven questions. The first question aimed to reveal students' mathematical discourses on infinity, and the remaining questions were targeted at investigating students' mathematical discourses on infinity and limit. Figure 1 shows a sample of interview questions.

The four Korean student study participants had been living in the US and attending US schools for more than three years. Participants within the same grade level (elementary, middle and high schools, and college) were selected based on the criteria of the same age, grade, and educational institution. For instance, the middle and high school students selected for the study attended the same schools in the same school district, and the undergraduates in both groups were enrolled at the same university. The pair of elementary school students attended different schools from each other. All students, except the undergraduate students in each group who had taken a calculus course, said that they had no formal education about mathematical infinity and limit. Because all Korean students had been living in the US and attending US school for more than three years, they could have been influenced by the colloquial and mathematical English discourses on infinity and limits.

I. Which is a greater amount and how do you know?

1. A: Your fingers — B: Your toes
2. A: Odd numbers — B: Even numbers
3. A: Odd numbers — B: Integers

II. What do you think will happen later in this table? How do you know?

x	$\frac{\sqrt{x+25}-5}{x}$
1.0	0.099020
0.5	0.099505
0.1	0.099900
0.05	0.099950
…	…

III. What is the limit of the following when x goes to infinity?

4. $\frac{1}{x}$ 5. $\frac{x^2}{1+x}$ 6. $\frac{x^2}{(1+x)^2}$

Figure 1: Representative samples of questions

The audio- and video-taped individual interviews lasted between 30 to 40 minutes per interview. The interviews were conducted in English and transcribed in their entirety.

As noted earlier, the reason for a comparison between American and Korean students is that the Korean terms for infinity and limit in a mathematical context rarely appear in colloquial Korean language. Therefore, while US students have experience with the colloquial use of the English words “infinity” and “limit”, Korean students have little experience with colloquial Korean use of the mathematical terms. Information about instructional materials in school curricula shows that Korean students had more intensively studied topics related to the words *infinity* and *limit* than US students.

Data analysis was conducted to identify and obtain detailed information on the distinctive features of routines in the two ethnic groups’ discourses. In this process, several comparisons were made: (a) for the two groups, we looked for salient characteristics of routines for each group; (b) we searched for similarities and differences between the groups’ uses of routines; and (c) we compared routines of the two ethnic groups based on linguistic differences.

Our sample size is too small to permit confident generalizations about the effects of linguistic differences on mathematics learning, but the results of our study can serve as a basis for hypotheses for future testing in a more comprehensive project.

FINDINGS

Infinity

In order to explore students' mathematical discourse on infinity, students were asked to compare pairs of infinite sets. They were also asked to compare a pair of finite sets so that we could determine whether they used the same routines for comparing finite and infinite sets. Table 1 shows the responses to other questions about comparing pairs of two infinite sets. In summary, the Korean students seemed to focus on elements of sets in their use of the words "odd numbers, even numbers, and integers" when comparing a pair of infinite sets, whereas US students compared the sets on the basis of entire sets. In addition, the difference in word use (a set-based approach in the US group versus an element-based approach in the Korean group) was related to differences in routines. The following section discusses when each routine is used and how each is implemented.

Table 1: Summary of comparisons between odd and either even numbers or integers

Students		Which is a greater amount and how do you know?	
		(b) A: Odd numbers, B: Even numbers	(c) A: Odd numbers, B: Integers
English	A5	[1] They [A and B] are the same because if an even number comes up, then an odd number comes up and an even number and odd, even, odd, even, so…	[5] They [A and B] are the same because odd numbers go up and integers keep going up and odd numbers and integers keep going up.
	A7	[2] They are equal because they both go on forever and once you had an even number then the next one is odd. So, there isn't really a place that they both end.	[6] I think it's integers because they can be every number...odd numbers are only a half of integers.
	A10	[3] They are the same because they are an infinite amount of numbers.	[7] A and B are equally…because all patterns of numbers...keep going on and on.
	AU	[4] I think the same because they are… numbers are infinite…so even…	[8] Odds are every other number…odd numbers are a part of integers and more integers.
Korean	K4	[9] Even numbers because odd numbers ties of odd numbers like nine like that. But the highest is even number like ten. Something like that.	[13] Integers because integers can be like any number but odd numbers can only be like odd numbers.
	K7	[10] I think they are equal because for every odd number, the next number is an even number.	[14] I think B because odd numbers are…there is only a half of the amount…integers are all like one, two, three, four…odd numbers are only one, three.
	K10	[11] You have to know where is the	[15] Integers are greater than odd

	end…even numbers can't be bigger than the odd numbers…odd numbers can't...	numbers in that sense because integers include odd numbers.
KU	[12] Even numbers because they start from zero but odd numbers start from one	[16] Integers because they also include even numbers.

Finite sets

When asked "Which is a greater amount and how do you know (between your finger and toes)?", almost all US and Korean students used the same number-word routine. They first counted ten (or five) fingers and ten (or five) toes and then compared the final number to get the answer.

Two exclusive infinite sets (odd and even numbers)

In the case of comparing odd and even numbers (sets with no common elements), there was a considerable difference between the US and Korean groups. The Koreans focused on individual elements as a means of comparing odd numbers with even numbers. Some students (K4 and KU) selected even numbers as having a greater amount because of either the starting [12] or the ending number [9]. Their explanations seemed to be based on a "competition" (or race) between two running lists of odd and even numbers ("the highest" [9], "the next number" [10], "the end" [11], and "zero" or "one" [12]). By using phrases like "the next number" [10] and "the end" [11], the other Korean students (K7 and K10) concluded that the amount of odd and even numbers is the same or incomparable.

The US students considered the entire sets of odd and even numbers when comparing them. The younger US students (A5 and A7) observed a one-to-one infinite correspondence between the two running lists ("…odd, even, odd, even, so…"[1], "there isn't really a place that they both end" [2]). They concluded that the amount of odd and even numbers is equal because of an operational use of the infinite sets, that is, as referring to an infinite process in the sets. The older US students (A10 and AU) said that the amount of odd and even numbers are equal because the sets have an infinite amounts of numbers ("an infinite amount of numbers" [3], "infinite…even" [4]). In both cases, the object of analysis for the US students is the entire set. Although the same number-word routine is not mathematically applicable to a pair of infinite sets, there were signs of attempts to adopt this routine to the infinite sets. A10 and AU noted: "They are the same because they have an infinite amount of numbers" ([3], [4]). This statement seems to indicate a belief that infinite and finite are interchangeable in terms of their properties. For instance, the students seem to imply that the word finite could be replaced with infinite in the sentence "Two finite sets are the same because they have the same number of elements."

Two inclusive infinite sets (odd numbers and integers)

When comparing odd numbers with integers, both groups used the part-to-whole routine; subjects noted that one set (e.g., odd numbers) is a part of the other set (e.g., integers) ([6], [8], [13], [14], [15], and [16]). All the Korean students and some of the US students (A7 and AU) who used the part-to-whole routine concluded that the number of integers is greater than the number of odd numbers. However, there is an ontological difference between the US and Korean students in using the part-to-whole routine. The Koreans seemed to suggest that odd numbers were a part of integers because the individual elements of integers include each odd number. By contrast, the US students who used the part-to-whole routine concluded that integers include odd numbers because of an operational use of elements within the sets of integers and odd numbers: the idea that integers are "every number," and odd numbers are "every other number" ("integers…can be every number…instead of just odd numbers" [6], "odds are every other number" [8]). The US students' use of the patterns "every number" and "every other number" suggests that their routines are based on the entire sets of odd numbers and integers rather than individual elements. The other US students (A5 and A10), who used the "race" analogy for comparing odd numbers with integers, concluded that the number of odd numbers and integers is the same because they both keep going on ("keep going up" [5], "keep going on and on" [7]). Once again, the subjects' routines can be interpreted on the basis of an operational use in the infinite sets of odd numbers and integers. Therefore, the US students used the two different routines grounded in the infinite sets of odd numbers and integers, whereas the Koreans employed the part-to-whole routine based on individual elements of odd numbers and integers.

Table 2 summarizes the features of the students' routines in comparing two infinite sets. It is noteworthy that all of the US and Korean students used different routines in the case of questions (b) and (c). For instance, all the Korean students who used the routine of comparison as a "race" in question (b) applied the part-to-whole routine to question (c). Thus, the comparison routines seemed to be highly context-dependent because the students in both groups used different routines in different cases.

Table 2: Summary of routines in comparison

Students		(b) odd and even numbers	(c) odd numbers and integers
American	A5	one-to-one correspondence in the comparison as a "race" based on infinite sets	routines of using an infinite going-up process based on sets
	A7		part-to-whole routine based on sets
	A10	routines of using an infinite amount based on sets	routines of using an infinite going-up process based on sets
	AU		part-to-whole routine based on sets
Korean	K4 K7 K10 KU	comparison as a "race" based on individual elements	part-to-whole routine based on individual elements.

Limit

Sequence

In order to provide material for the investigation of the mathematical discourse on infinity and limit, students were asked to find the limit of an infinite sequence and to justify their answers. The questions: "What will happen later in this table? How do you know?" were used to investigate students' conceptions of the limit value of a sequence without using the word limit. When finding a pattern in the context of a given infinite sequence, the US 7th and 10th graders looked at either an increasing or decreasing pattern of numbers in the infinite sequence ("smaller and smaller", "keep increasing"). Their discourse seemed to be mediated syntactically, as it was based on changes in the sequence without the concept of limit. By contrast, the Korean 7th and 10th graders seemed to first look at the operational patterns of the infinite sequence and then implicitly present the number that those patterns approached as the limit value ("keep going up . . . to infinity", "keep get and go down . . . it will just come to one"). The two undergraduates mentioned that the pattern of 0.0999 . . . in the infinite sequence approached a value of 0.1 as its limit. The common characteristic is that they (AU, K7, K10, and KU) explicitly or implicitly objectified the operational patterns with the concept of limit as a number. The transition from syntactic to objectified mode suggests a certain degree of flexibility in their discourses. Their routines are more applicable to the task of finding limits in infinite sequences than those of A7 and A10.

Functions

In order to elicit students' mathematical discourse on infinity and limit, we asked them to find the limit of a given function when x goes to infinity. Table 3 summarizes students' responses about the limit of $1/x$. When students were asked to calculate the limit of an infinite sequence and with a function, all of the Korean students (except the elementary student) and the US undergraduate showed different patterns from the US 7th and 10th graders. The prevalent features of the students' routines were deeply related to their word use characteristics for infinity and limit: an operational use and a number-based operation use.

Table 3. Summary of response about the limit of 1/x

Students		(b) What is the limit of $1/x$ when x goes to infinity?
American	A5	[1] . . . x equals…infinity approaches . . . it's like zero.
	A7	[2] I don't know what the limit is . . . That's gonna be one over infinity. When x goes on forever, one over forever.
	A10	[3] The limit is always one over infinity . . . infinity does have no limit … something keeps going on and on . . . I am not sure what the limit is.
	AU	[4] One (problem) would be zero because the bigger x gets . . . it just be smaller and smaller decimal. It's a sense of going infinity, there is no ending which is gonna be very, very tiny.
Korean	K4	[5] I don't get what it means by when x goes to infinity…
	K7	[6] This would become zero . . . because the number one, you know one…this will be close to thirty . . . It won't really be anything.
	K10	[7] It will get close to zero…because x is two…point five…x…ten…it will get to keep smaller and . . .
	KU	[8] One over x goes to infinity…then I can say like…then I can think like one over infinity . . . if I divided one…one over one…but one over two is point five . . . that meaning is zero because infinity means too large number.

In calculating the limit in the case of $1/x$, the elementary students also seemed to have no idea of the given context. The routines of the US 7th and 10th graders were to substitute infinity for x by alluding to an operational use of infinity ("when x goes on forever" [2] and "something keep going on and on" [3]). Then, in the more complicated functions

$x^2/(1+x)$ and $x^2/(1+x)^2$, their routines for finding the limit of each function was the same as the routine in the case of (“infinity squared over one plus infinity” by A7, “infinity is being multiplied by itself” by A10, “infinity divided by itself” by A10). They seemed to be using just one mediational mode (syntactic) with little flexibility. This syntactic mode allows for little interpretation of finding limits and no predictions.

By contrast, the Koreans (K7, K10, and KU) and the US undergraduate substituted several increasing numbers for x (“the bigger x gets” [4], “the number one, you know one . . . this will be close thirty” [6], “x is two . . . point five . . . x . . . ten . . .” [7], and “if I divided one . . . but one over two . . . infinity means too large number” [8]) and checked whether the values of $1/x$ were increasing or decreasing to determine the limit. Then, in the more complex cases of $x^2/(1+x)$ and $x^2/(1+x)^2$, the Koreans used the routine of deciding whether numerators are bigger than denominators to find the limit (“x squared always is bigger than 1 plus x” by K7, “x is one . . . one . . . two, x is two then four . . . nine . . . so it will get bigger . . .” by K10, “infinity square is much bigger than one plus infinity” by KU). Unlike the Korean students, the US undergraduate considered whether the entire values of each function are increasing or decreasing to determine the limit.

In the case of $1/x$, they (K7, K10, KU, and AU) made a few transitions from one mediational mode to another; from syntactic to concrete mode and then from concrete to objectified mode. In the cases of $x^2/(1+x)$ and $x^2/(1+x)^2$, the Koreans (K7, K10, and KU) made an additional transition to a new *objectified* mode, as they created the *numerator-denominator-comparison* routine rather than using substituted numbers in their mathematical discourse. Their flexibility from one mediational mode to another in the process of finding limits can be described as *objectified* discourse. This mediational flexibility provides more interpretations and predications regarding the concept of limit than only one mediational mode. Based on mediational flexibility, Table 4 summarizes the characteristics in the students’ routines in the two situations (i.e., when calculating the limit of an infinite sequence and calculating the limit with a function).

Table 4: Summary of mediational flexibility in calculating limits

<table>
<tr><th colspan="2">Students</th><th>(a) sequence</th><th>(b) $1/x$</th><th>(b) $x^2/(1+x)$</th><th>(b) $x^2/(1+x)^2$</th></tr>
<tr><td rowspan="4">American</td><td>A5</td><td colspan="4">No idea</td></tr>
<tr><td>A7</td><td rowspan="2" colspan="4">Syntactic mode</td></tr>
<tr><td>A10</td></tr>
<tr><td>AU</td><td>from syntactic to objectified mode</td><td colspan="3">from syntactic to concrete mode;
from concrete to objectified mode</td></tr>
<tr><td rowspan="4">Korean</td><td>K4</td><td colspan="4">No idea</td></tr>
<tr><td>K7</td><td rowspan="3">from syntactic to objectified mode</td><td rowspan="3" colspan="3">from syntactic to concrete mode;
from concrete to objectified mode</td></tr>
<tr><td>K10</td></tr>
<tr><td>KU</td></tr>
</table>

DISCUSSION

Routines based on word use

The first research aim pertains to the characteristics of students' mathematical discourses on infinity and limit in terms of routines. In the mathematical discourse, one important characteristic of the US students was a set-based approach to infinity, which means that they focused on an operational use of infinity rather than on numerical values. The set-based word use was present in two different forms: the set of all numbers in cardinality comparisons and an operational use in calculating limits (cf., a numerically-based operational use by AU). The first word use is set-based because comparisons are related to an operational use of elements within sets of numbers. In the second use, students substituted infinity as a process for x to find the limit for a given function. There was no dependency on numerical values. Although the US undergraduate used set-based words in the cardinality comparison, she employed element-based word use (a number-based operational use) in calculating limits, unlike the other US students (grades 7 and 10). This difference may be related to her formal education on limit.

One noticeable common characteristic of the word use of the Korean students in the mathematical discourse on infinity was their predominantly element-based approach. Element-based word use is characterized by a focus on numbers themselves rather than on sets of numbers. This word use was observed in two different mathematical tasks: individual numbers in cardinality comparison and a number-based operational use of infinity in calculating limits. The first such word use is element-based, since individual numbers (elements) were compared rather than their respective sets (odds, evens, and integers). In the second task, students considered several increasing input values (elements) for each function in order to determine the limit.

As for the second research aim about salient differences between the Korean and US groups, the different characteristics in word use are with routines. For instance, in the cardinality comparison task, the element-based use of the words "odd numbers, even numbers, and integers" seemed to lead the Korean students to use either a "race" comparison routine or a part-to-whole routine, both of which are based on relationships between individual elements of these groups of numbers. In the calculating limit tasks, an element-based word use (i.e., a number-based operational use of infinity) was related to the flexible use of visual mediators (involving both concrete and objectified modes). To calculate the limit of each function, manipulation of concrete input values (a number-based operational use of infinity) was related to both the substituted values of each function (*concrete* mode) and an approachable number by increasing or decreasing these values (*objectified* mode). This also shows a flexible transition from one mediational mode to another in routines.

Comparatively, in the mathematical discourse of cardinalities of sets, the set-based use of the words "odd numbers, even numbers, and integers" seemed to lead the US students' routines, which were also based on the comparison of entire sets. In limit-finding tasks, an operational use of infinity was related to only one mediational mode (syntactic) in the use of visual mediators because infinity was scanned as a process and the variable x in each function was replaced by infinity. This shows little flexibility in routines. Based on the observations of students' routines, the US and Korean students displayed qualitatively different patterns grounded in their word use in different cases.

A salient property in routines is that all students used different routines for the different cases. It is noteworthy that all four Korean students had different routines for the different cases. This implies that their routines are highly context-dependent. For instance, they used a "race" comparison routine to compare odd and even numbers, but applied a part-to-whole routine in comparing odd numbers with integers. This may be explained on the basis of the fact that learning (transfer) does not happen automatically, but takes time and practice. Although more information is needed to fully understand students' meta-rules, routines seem to be highly context-dependent.

Impact of linguistic differences on routines

One significant characteristic of US students' colloquial discourse on infinity and limit is their use of real-life contexts and an operational use in their application. By contrast, the Korean group's explanations were more abstract and mathematical, and they used the words structurally rather than operationally (Kim, Sfard, & Ferrini-Mundy, 2010). Students' colloquial discourse about the notions may thus have an impact on their later mathematical *word use* and other aspects of their mathematical discourse, such as *routines*. This is evidenced by the application of the word *infinite* to numbers (the *elements* of sets in the Korean group versus the *sets* of numbers in the US group) in colloquial discourse related to the two characteristics (the *element-based approach* in the Korean group versus the *set-based approach* in the American group) in mathematical discourse. Thus, colloquial discourse seems to have an impact on mathematical discourse because of certain clear differences between the mathematical discourses on infinity and limit between US and Korean students; these differences may be ascribed to the mathematical words infinity and limit not being available in the Korean language colloquial discourse.

Other factors apart from language may account for the observed differences between the discourses of the Korean-speaking and English-speaking students. For example, Korean students could be more advanced in their formal mathematical discourse on infinity and limit because they had slightly more opportunities to learn about infinity and limit in their formal school setting. However, since English-speakers' discourse on infinity and limit tends to be mainly processual, these students need help to develop their objectifications in the discourse. Since the Korean-speakers' discourse is more abstract and formal, both structural and element-related approaches to infinity and limit have to be attended to with much care.

CONCLUSIONS

Understanding students' routines can give mathematics educators insights into mathematics education and provide solutions to unresolved learning difficulties through understanding the role of language in learning. Mathematics learning and routines are inextricably interwoven. In addition, the linguistic infrastructure of mathematical discourse can be responsible for the differences in routines. For example, little continuity in lexical development between Korean colloquial and mathematical discourses can account for Korean students' abstract and structuralized word uses and routines. By contrast, US students' mainly processual word uses and routines can be ascribed to continuity in lexical development between English colloquial and mathematical discourses. Thus, the results of this study imply the different needs of Korean and US students in their discursive development.

There is a need to move beyond cognitive methods of research to uncover contextual sensitivity in mathematical learning, such as the use of language, the active process of “enculturation into a community of practice” (Cobb, 1994), and continuous changes of learning contexts. Without examining learning context, such as linguistic differences, researchers may obtain distorted findings. To reveal situated learning difficulties, researchers need to investigate thinking (a process of learning) in context rather than thought (the second-construct of thinking). The multi-lateral approach of discourse analysis is a theory-based method emphasizing contextual sensitivity to the use of language. It is a promising research method to reveal the mechanisms of mathematical learning in complex contexts because it can deepen our fundamental understanding and provide pragmatic processes to resolve student learning difficulties.

REFERENCES

Borasi, R. (1985) Errors in the enumeration of infinite sets. *Focus on Learning Problems in Mathematics, 7,* 77–89.

Cobb, P. (1994) Where is the mind? Constructivist and socio-cultural approaches on mathematical development. *Educational Researcher*, *23*, 13–20.

Cornu, B. (1992) Limits. In D. Tall (ed) *Advanced mathematical thinking* (pp. 153–166). Dordrecht, The Netherlands: Kluwer Academic.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., and Vidakovic, D. (1996) Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior, 15*, 167–192.

Davis, R., & Vinner, S. (1986) The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior, 5,* 281–303.

Fischbein, E. (2001) Tacit models and infinity. *Educational Studies in Mathematics*, 48, 309–329.

Fischbein, E., Tirosh, D., & Hess, P. (1979) The intuition of infinity. *Educational Studies in Mathematics, 10,* 3–40.

Hersh, R. (1997) *What is mathematics, really?* New York, NY: Oxford University Press.

Kim, D. (2010) The histories of the mathematical concepts of infinity and limit in a three-fold role. *The Journal of Educational Research in Mathematics, 20*(3), 293–303.

Kim, D., Sfard, A., & Ferrini-Mundy, J. (2010) Students' colloquial and mathematical discourses on infinity and limit: A comparison of U.S. and Korean students. *School Mathematics, 12*(1), 1–15.

Nunes, T., Schliemann, A., & Carraher, D. (1993) *Street mathematics and school mathematics*. Cambridge, MA: Cambridge University Press.

Przenioslo, M. (2004) Image of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics, 55*, 103–132.

Rogoff, B. (1990) Apprenticeship in thinking: Cognitive development in social context. New York, NY: Oxford University Press.

Sfard, A. (2008) Thinking as communicating: Human development, the growth of discourses and mathematizing. New York, NY: Cambridge University Press.

Tall, D. (1992) The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. A. Grouws (ed) *Handbook of research on mathematics teaching and learning* (pp. 495–5 I 1). New York, NY: Macmillan.

Tirosh, D. (1992) The role of students' intuitions of infinity in teaching Cantorian theory. In D. Tall (ed) *Advanced mathematical thinking* (pp. 199–214). Dordrecht, The Netherlands: Kluwer Academic.

Tsamir, P. and Dreyfus, T. (2002) Comparing infinite sets – a process of abstraction: The case of Ben. *Journal of Mathematical Behaviour, 21*, 1–23.

Vinner, S. (1992) The role of definitions in the teaching and learning of mathematics. In D. Tall (ed) *Advanced mathematical thinking* (pp. 65–81). Dordrecht, The Netherlands: Kluwer Academic.

Vygotsky, L. S. (1978) Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986) *Thought and language.* Cambridge, MA: MIT Press.

Weller, K., Brown, A., Dubinsky, E., McDonald, M., & Stenger, C. (2004) Intimations of infinity. *Notices of the American Mathematical Society*, *51*, 741–750.

Williams, S. R. (2001) Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education*, *32*(4), 343–367.