Threshold concepts and troublesome knowledge in a second-level mathematics course

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Introduction

The term threshold concept derives from education theory to denote concepts that are essential to knowledge and understanding within particular disciplines. Threshold concepts act like doorways that once crossed enable students to comprehend a topic not previously understood. In turn this enables the learner to progress to higher levels of learning. This process can be immediate or drawn out over long periods of time. Grasping the threshold concept transforms the students' perception of the subject area, and they are better able to relate the topic to wider fields of study (Meyer and Land 2003). However the transformative nature of this process can at times be troublesome and challenging for students (Perkins 1999). In mathematics, once a threshold concept is grasped, the student will see the calculations they are working on in a different light. They will gain insight into what the calculations are doing and how they work. This enables the student to translate the threshold concept to different and more difficult problems. The student is also able to see the relevance of this form of mathematical thinking to other areas of mathematics and to applications beyond the field of mathematics.

Very little work has been undertaken to identify the threshold concepts in higher-level mathematics and what is troublesome for students to learn. This paper reports on a study carried out in a large second-level mathematics course at the University of Queensland. The course is taken by students of mathematics, engineering and physical sciences, and covers topics such as advanced ordinary differential equations, vector calculus and linear algebra. Data was collected from course documents and interviews with tutors, and surveys and quizzes completed by students. Analysis of this data identified potential threshold concepts for this content area, and areas of troublesome knowledge experienced by students. This paper reports on the findings from this study, and on the implications of these findings for enhancing learning and teaching of mathematics.

Course content

The first part of the course consisted of three separated topics: linear second-order ordinary differential equations (ODE), hyperbolic functions and multiple integrals. The initial phase of the study was to analyse the relevant course material, including lecture notes, tutorial sheets and assignments, and use tutor interviews to obtain some insight into the troublesome areas.

Students had seen some of the techniques for solving ODEs, including the method of undetermined coefficients, in a previous course. These were revised at the start of the course and further techniques were introduced in lectures, including reduction of order and variation of parameters. The tutors reported that the students had difficulty understanding the new techniques and needed to revise them in the tutorial classes. Students also had difficulty with the one tutorial question that involved an application and wordy explanation but were able to solve other ODEs by using the techniques. The assignment involved testing the students on their ability to use the techniques rather than on their understanding of what they were doing, and in this most students did very well.

The second topic in the course was a short introduction to hyperbolic functions, providing various properties and identities to give students a good grounding in their use. However, many students failed to see the relevance of this section, as it appears very isolated from the rest of the course, and did not appreciate that hyperbolic functions and their inverses are used in many questions that they would later encounter. The catenary problem was the only application shown in lectures but it was not used in either tutorial sheets or assignments. The tutors reported that students were having difficulty with questions that involved hyperbolic function substitution. In particular, students were not sure which function they should use as a substitution in order to solve the integral.

It is useful to identify this problem as in other parts of the course students will have to use substitution techniques. Inclusion of this section is necessary to gain knowledge about hyperbolic functions and revise algebraic techniques. It is not difficult or troublesome in other ways.

This section brings students on to multiple integrals, which introduces a new dimension into their work. It is important at this stage for students to be strong in the techniques they have previously used in single integration in order to translate these to triple integrals. The first part of the section deals in double integrals over rectangular shapes and moves on to other general regions. Some of these regions need to be broken up into smaller units to enable integration of the region while others need their order of integration changed to make to make easier or sometimes possible to integrate. For double integration, tutors reported that students found working out the boundaries and changing the order of integration the most troublesome. Many students had trouble converting rectangular coordinates to polar coordinates and some avoided using them, making their calculations more difficult. The tutors attempted to overcome this reluctance by working through these questions with students in tutorials. The troublesome areas in triple integrals were similar. Because they were triple integrals this compounded the difficulty, especially when using cylindrical and spherical coordinates. All of these reflect the conceptual difficulties that students are having with multiple integrals.

Quiz

In the first semester of the study, students engaged in a short quiz held on the material from this first section of the course. Though the lectures had moved on to the next section, the students had recently completed a take-home examination and several assignments on this material. The quiz was designed to test both conceptual understanding and procedural knowledge-based understanding. The conceptual understanding questions were in an open format to allow a variety of answers. The knowledge and procedural knowledge questions tended to be in a true or false format.

The quiz was held at the beginning of the lecture presented on thirteen slides. These were shown over a period of ten minutes while students answered the questions on the answer sheet provided. When the majority of students stopped writing a new slide was presented. Not all students arrived on time and the later students only completed the remaining questions. At the end of the quiz, the answer sheets were collected and the normal lecture started. A total of 42 students answered at least one question on the quiz.

The responses to each open question were split into four categories of high understanding, understanding, low understanding and no understanding. The responses were categorised and checked by another tutor in the course, with discussions on any disagreements on categorising. The categories were also given numerical values, from 0 (no understanding) to 3 (full understanding), for calculating marks. Questions with true or false answers were given one mark for the correct answer and no marks for the wrong answer. The remaining questions were given a mark for each correct answer and no mark if incorrect. The marks were then added together to give a mark out of 40 and then scaled to give a mark out of 50, to match the range of the mid-semester examination and final examination. The results discussed below come from the 33 students who were at the quiz for the total duration and supplied us with their student number (for comparison with the examination results).

From this data, 43% of students indicated that they did not know what an ODE was, with 55% not understanding what they are looking for when they solve an ODE and another 40% not knowing what we are referring to when we call them second order. Most of the students (73%) were able to match up the roots of the characteristic equation with the solution but the same percentage of students did not know why we use reduction of order. For the statements regarding nonhomogeneous second-order ODEs, 73% of the students were correctly able to indicate all the ones that were true or false whereas 31% were able to do this for the statements on the method of variation of parameters. Only 31% were able to correctly answer the corresponding questions for hyperbolic functions. On the question on multiple integrals, 58% of students were able to put the correct values in the bounds of integration. Most students (61%) were not able to identify the only possible way to do the integral given but (79%) were able to recognise the most suitable types of coordinates to use when they are asked to find the volume of a solid bounded by a plane and a paraboloid. In multiple integrals, very few students (6%) were able to answer all the true or false questions correctly on the statements regarding the bounds of integration. The last question gave four techniques and asked which ones can be used when evaluating a triple integral. Of these 21% of students recognised that they could use all of them.

Table 1 compares the results from all the students who took the quiz to their results in the mid-semester and final examinations. All the results are out of 50 for each examination.

Table 1. Comparison of the quiz with mid-semester and final examinations

	Quiz	Mid-Semester	Final Examination
Average (/50)	27.99	40.26	29.56
SD	7.62	6.93	9.99

There were significant correlations between the quiz and the mid-semester examination ($r = .53$, $p < .05$), between the quiz and the final examination ($r = .57$, $p < .05$), and between the mid-semester examination and the final examination ($r = .37$, $p < .05$).

The purpose of this quiz was to test the student's conceptual understanding of the course. The results in the first seven questions suggest that the students do not conceptually understand the techniques used in solving linear second-order ODEs. However, they did show good ability at using the methods. There was one knowledge question on hyperbolic functions, which indicated that students had reasonable knowledge in this area. The remaining questions, requiring a mixture of conceptual understanding and procedural knowledge, referred to multiple integrals. Students showed a variety of understanding of parts on both their conceptual understanding and procedural knowledge.

Overall it seems students are generally relying on their ability to use the methods given in the course rather than conceptually understanding what they are doing. The mid-semester was a take home exam, which would explain why the mean and standard deviation for the quiz might be closer to the final examination than the mid-semester examination. Although the results were a lot higher in the mid-semester exam, there was still a significant (positive) correlation with the quiz. However, the final examination for these students was closer to the quiz and had a higher correlation with the results. This would indicate that the final examination might also act as a good indicator of conceptual understanding.

Survey

In the second semester of this study there were approximately 750 students, spread over two lecture streams, in contrast to the single stream of 146 enrolled students when the above quiz was used. This made the quiz impractical and so a survey was used instead. Students were given the survey at the beginning of lectures with three questions asking ‘what did you find difficult or troublesome’ in the sections onlinear second order ODEs, hyperbolic functions and multiple integrals. These were collected at the end of the lectures. Out of approximately 500 students present at the two lecture streams, 124 completed the survey.

Over half of the students (56.5%) indicated they had no problems with the linear second-order ODEs topic. However, fewer students found the sections on hyperbolic functions (33.9%) and multiple integrals (29.8%).

Table 2 shows the three areas in ODEs that students found most difficult or troublesome. For this topic the main difficulties were with the method of undetermined coefficients, variation of parameters and reduction of order. Very few students (< 3.5%) had difficultly in other aspects.

Table 2. Difficulties with ODEs

Difficulty	%
Method of undetermined coefficients	15.3
Variation of Parameters	11.3
Reduction of Order	9.7

Table 3 similarly shows the five areas in hyperbolic function that students found most difficult or troublesome. Nearly ten percent of students indicated that they had general difficulty with this section, with specific difficulties in substitution, differentiation and integration, and definition and identities. Less than five percent of students indicated they had difficulty with other aspects.

Table 3. Difficulties with hyperbolic functions

Difficulty	%
Substitution	17.7
Differentiation and integration	9.7
General difficulty	9.7
Definitions and identities	8.1

Finally, Table 4 shows the eight areas that students found difficult or troublesome with multiple integrals. The same ten percent of the students indicated that they had general difficulty with multiple integrals. In individual areas, main difficulties included limits of integration, changing order of integration and converting coordinates. Other issues were with visualising three-dimensional graphs and calculating the domains of integration. Less than five percent of students had difficulty with areas other than these.

Table 4. Difficulties with multiple integrals

Difficulty	%
Limits of integration	21.8
Order of integration	21.0
Converting Coordinates	19.4
General difficulty	9.7
Visualising 3D graphs	8.9
Domains of integration	7.3
Integration	5.6

The purpose of the survey was to establish what students found troublesome or difficult in these sections of the course and thus ascertain what, if any, are the threshold concepts.

Discussion

This study set out with the intention to identify threshold concepts and find out what material students found troublesome in the course. As Eckerdal, McCartney, Moström, Ratcliffe, Sanders and Zander (2006) note, threshold concepts act like boundary markers that define where a subject starts or end, and so can be useful to identify. There are three potential threshold concepts identified from the information gathered, supported by the above quiz and survey results and interviews with tutors.

The first candidate threshold concept is an ordinary differential equation. Techniques for solving ODEs appeared to be a problem for many students. The tutors had identified variation of parameter and reduction of orders as particularly problems and the students added the method of undetermined coefficients through the survey. However, even though the quiz indicated conceptual difficulty with ODEs, the mid-semester examinations showed strength in applying the methods. Therefore, ODEs are conceptually difficult to understand and contain elements of knowledge that students find troublesome. Understanding ordinary differential equations continues beyond this course to third and fourth year mathematics subjects, with each piece of knowledge gathered on ODEs building on techniques and understanding from the previous course. Fully grasping the threshold concept occurs along this journey. Some students learn early on, enabling them to apply what they have learnt to later courses, while others take time and struggle as they go. Many of the students in this course appear to be in a state of 'liminality' (Meyer and Land 2003) or partial comprehension; using the techniques they have learnt to solve the equations without fully understanding what they are doing.

A second threshold concept candidate is the technique of substitution. Many students struggle with hyperbolic functions when first introduced to them in this course, and both students and tutors identified the method of substitution as the main difficulty. This is reflected by low scores on the mid-semester and final examinations. Though hyperbolic functions are troublesome they are not transformative, suggesting that they are not a threshold concept in themselves. In contrast, the technique of substitution is useful throughout calculus, beyond the context of hyperbolic functions in the course. For example, when changing coordinates in multiple integrals students identified the method of substituting one coordinate system for another as troublesome. The substitution technique is integrative and is in itself conceptually difficult.

The final candidate for a threshold concept is multiple integration. Here difficulties include changing from one coordinate system to another, working out the limits of integration and changing the order of integration. The tutors echoed these difficulties, particularly with triple integrals as opposed to double integrals. Students performed well in solving double integrals in both the mid-semester and final examination but did not do so well on a triple integral question. Triple integration moves on from double integration as it introduces another dimension in the students' work. Students identified visualising three dimensions as one of their difficulties. Their responses to the survey indicate it is troublesome in many ways and conceptually difficult to understand. It is also integrative with other subjects and not just mathematics. Once understood the students will be able to apply their understanding to other triple integral problems and applications not seen in this course. Further investigation is needed to verify the threshold concepts and troublesome knowledge identified and seek other candidates from the later section of the course

Conceptual understanding verses procedural knowledge

The course structure provides students with plenty of opportunities to practice procedures and gain conceptual understanding of the course material. The lectures show how to derive the course material as opposed to just giving the procedures. The tutorial sheets and assignments also provide students with practise of procedures and give questions that are conceptually difficult. In addition, students are able to attend tutorials for help with difficult questions. These give students an opportunity to see some of the course material presented differently than seen in lectures. Furthermore, the lecturers provide consultation hours for getting help and an open forum on the course website to ask questions. In this way, the course uses both active and social learning. Not all students gain conceptual understanding of the course material regardless of the opportunities presented as indicated by the quiz and tutors experience. For example, many tutors found that the majority of students did not attempt tutorial sheets before the tutorial class.

There are difficulties in keeping students motivated to come to tutorials. Tutors found that when assignments were due numbers would increase but would then dramatically decrease at other times. After submission of the last assignment, some tutors found few or no students in their class. The tutors are an essential element of the course in gaining these threshold concepts, bridging the gap of conceptual understanding not gained in lectures. Helping the students with assignments in tutorial classes takes away the opportunity for students to practise many parts of the course. One solution would be for assignment help to be withdrawn from tutorials and provided instead within informal but supported learning settings (e.g. the first year learning centre). Also making tutorial attendance assessable may prevent numbers reclining in tutorials. Many first year classes make attending tutorials part of the assessment.

Implications

This study enabled areas of conceptual difficulty in the course to be identified and possibly addressed in future courses and tutorial classes. The tutors provided the study with invaluable information that reconfirmed the troublesome areas student were experiencing. During these interviews, it became apparent that the tutors found these conversations to be a valuable and enjoyable addition to their teaching experience. In particular they found these sessions provided opportunities to debrief with each other, to discuss what went well in their classes, as well as seek advice. Another outcome of this study is to consider the potential of ongoing mentoring sessions for tutors.

References

Eckerdal, A., McCartney, R., Moström, J. E., Ratcliffe, M., Sanders, K. and Zander, C. (2006) Putting threshold concepts into context in computer science education. In *ITiCSE-06* (103–107). Bologna, Italy.

Meyer, J. and Land, R. (2003) Threshold concepts and troublesome knowledge: linkages to ways of thinking and practising within the disciplines. In C. Rust (Ed.), *Improving student learning theory and practice - 10 years on* (412–424). Oxford: OCSLD.

Perkins, D. (1999) The many faces of constructivism. *Educational Leader,* **57**(3), 6–11.

Worsley, S., Bulmer, M. and O'Brien, M. (2008) Threshold concepts and troublesome knowledge in a second-level mathematics course. In A. Hugman and K. Placing (Eds) *Symposium Proceedings: Visualisation and Concept Development*, UniServe Science, The University of Sydney, 139–144.