THE USE OF INFLUENCE LINES IN THE DESIGNING OF STRUCTURES.

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(*A paper read before the Sydney University Engineering Society, August 12th,* 1903.)

THIS paper proposes to deal with a method, which has not hitherto had extensive application, of finding stresses in structures. This statement applies to works in English which have come under the writer's notice.

The use of Influence Lines in connection with various *structures will be elaborated, some examples will be given of their application, and their utility pointed out in comparison with other methods.

An Influence Line is a curve drawn such that its ordinate at any point represents the value of some function (such as a Bending Moment, Shear, or Reaction) when some unit load is in the position indicated by the abscissa.

Put into mathematical language, an Influence Line is the curve $y = f(x)$, where y represents the stress being investigated, and x the position on the structure.

The above definitions apply to loads moving on the horizontal and to a plane curve; this exists in the great majority of cases: however, it will be seen that the method could also be used for curves traced on a solid, where $f(x, y, z)$ may stand for one or more equations, but for the present we confine ourselves to plane curves, as in most cases it is a plane section of the structure which is considered for the investigation of stresses.

It will be seen how the Influence Line method may be applied. For the investigation of maxima and minima effects we can find the points by means of the Differential Calculus, and thus see where loads have the greatest or least positive and negative values.

By drawing the curve we can see the effect of partial loading with concentrated loads, as we need only multiply each load by the corresponding ordinate in proper units to get the effect due to that load. For distributed loads we can find the area of the curve between the limits of loading by means of the Integral Calculus (or by the planimeter or other approximate method) and thus deduce its effect.

The method may be used both for the discovery of the effects of certain loading and for the plotting of results already found; the two processes may go on at the same time.

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* "Structure" is used for truss, beam, arch, or whatever shape is under consideration.

For structures where there are a few crucial points (*c. f.* an arch without hinges) the method is particularly happy.

The writer hopes to illustrate these and other points during the course of the paper, and will endeavour to shew that the investigator has at his disposal an elegant method of considerable power.

In all cases, it is necessary, as is usual in tracing any curve, to note positions of symmetry, also where sudden changes occur, and the meaning of a negative sign.

The Influence Line curves must not be confused with ordinary Bending Moment and Shear diagrams, although in some cases they are similar in shape.

Case 1.—CANTILEVER (Fig. 1).

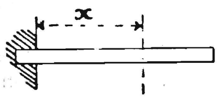


Fig 1

Let x = distance of load from abutment.
B. Mt. at Abut. = Wx where W is unity.
Influence Line of B. Mt. at Abut. is $y = x$. (Fig. 1*a*).
Influence Line Shear at Abut. v = unity. (Fig. 1*b*).

Referring to Fig. 1*a* the B. Mt. due to a distributed load of w per unit of length = area of triangle OPM = $\frac{1}{2} wx^2$.

Referring to Fig. 1*b* the Shear due to a distributed load of w per unit of length = area of rectangle OP = wx.

These results agree with those found in any other way. If we had a load W at the point M and a uniform load w per unit of length from A to B then B. Mt. at Abut. due to these loadings.

$$= \{W \times PM\} + \{\text{Area AKLB}\}\ w, \text{ all in same units.}$$

The influence line of B. Mt. at any other point is the straight line drawn at an angle of 45° with the horizontal at the point.

Influence line of shear at any point is a horizontal line drawn at a height unity above the point.

Case 2.—BEAM SUPPORTED AT BOTH ENDS (Fig. 2).

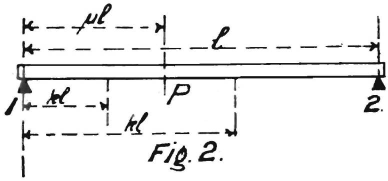


Fig. 2.

In the designing of a truss divided into panels, it is necessary to know the bending moments and shears at the panel points.

Let P be point considered.

Let μ = ratio of distance of point under consideration to total length of beam. kl = distance of unit load from 1.

(*a*) Load on P1

B. Mt. $= k \times (1 - \mu)\, l$.

Shear looking to right of P $= +\, k$.

" " " left of P $= -1 + (1 - k) = -\, k$.

(*b*) Load on P2

B. Mt. $= (1 - k) \times \mu l$.

Shear looking to right of P $= -\, (1 - k)$.

" " " left of P $= +\, (1 - k)$.

The sign of shear is taken positive when it has the same direction as the reaction.

From the above it is clear that the influence line of B. Mt. at a point P distant μl from one end is two straight lines drawn to a point at a vertical distance $\mu\,(1 - \mu)\, l$ above the point P. (Fig. 2*a*). Influence line for shear at P is two straight lines drawn to points μ and $(1 - \mu)$ above and below the horizontal at P or *vice-versa*. (Fig. 2*b*).

We can tabulate $\mu\,(1 - \mu)$; on account of symmetry, it is not necessary to go beyond $\mu = \frac{1}{2}$. (Table is for l = unity). When μ does not reduce to a decimal the second part of table is to be used.

*TABLE 2*a*.—Values of $\mu\,(1 - \mu)$.

(1) μ in decimals (Tenths in column, hundredths in row).

μ	0	1	2	3	4	5	6	7	8	9
·0	0	·010	·020	·029	·038	·048	·056	·065	·074	·082
·1	·090	·098	·106	·113	·120	·128	·134	·141	·148	·154
·2	·160	·166	·172	·177	·182	·188	·193	·197	·202	·206
·3	·210	.214	·218	·221	·224	·227	·230	·233	·236	·238
·4	·240	·242	·244	·245	·246	·247	·248	·249	·250	·250

(2) μ in fraction (Numerator in column. Denominator in row).

μ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	·250	·222	·188	·160	·139	·122	·109	·099	·090	·083	·076	·071	·066	·062	·059	·055	·052	·050	·048
2		0	·222	·250	·240	·222	·204	·188	·173	·160	·149	·139	·130	·122	·116	·109	·104	·099	·094	·090
3			0	·188	·240	·250	·245	·234	·222	·210	·198	·188	·178	·168	·160	·152	·145	·139	·133	·127
4				0	·160	·222	·245	·250	·247	·240	·231	·222	·213	·204	·196	·188	·180	·173	·166	·160
5					0	·139	·204	·234	·247	·250	·248	·243	·237	·230	·222	·215	·208	·201	·194	·188
6						0	·122	·188	·222	·240	·248	·250	·249	·245	·240	·234	·228	222	·216	·210
7							0	·109	·173	·210	·231	·243	·248	·250	·248	·246	·242	·238	·233	·227
8								0	·099	·160	·198	·222	·237	·245	·248	·250	·249	·247	·244	·240
9									0	·090	·149	·188	·213	·230	·240	·246	·249	·250	·249	·247
10										0	·083	·139	·178	·204	·222	·234	·242	·247	·249	·250

Some examples will be given of the use of these diagrams.

(*a*) *Taking the case of a hog-backed girder as a general case. Fig. 2c.*

Let M_1 = B. Mt. at P_1

S_1 = Shear at P_1

* Tables are numbered to agree with the Diagrams.

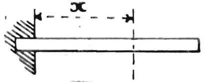


Fig 1

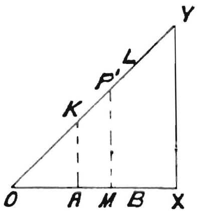


Fig. 1a B.M at Abutment.

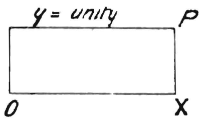


Fig. 1b. Shear

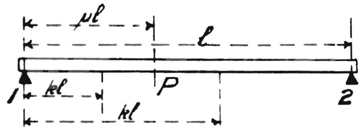


Fig. 2.

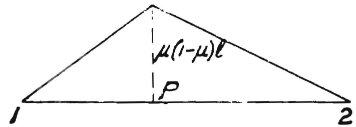


Fig. 2a. B.M.t at P

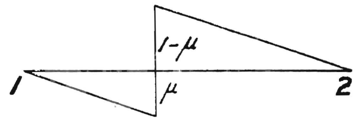


Fig. 2b. Shear at P
Reading to Abut 1.

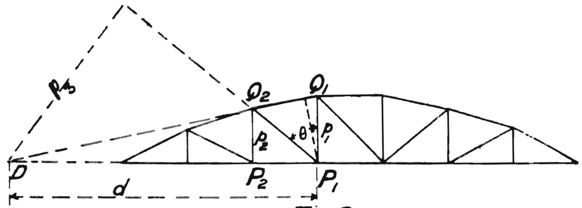


Fig. 2c.

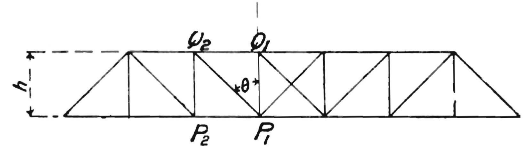


Fig. 2d.

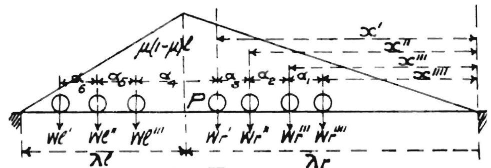


Fig. 2 e

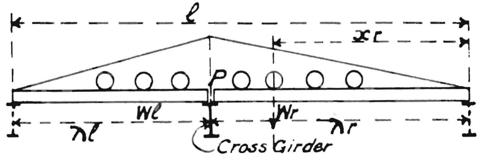


Fig. 2f.

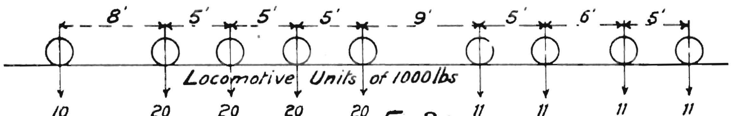


Fig. 2g.

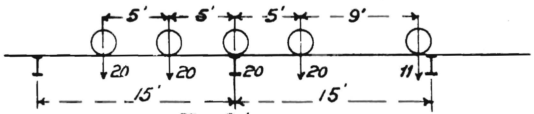


Fig. 2h.

Assume B. Mt. as positive when in same direction as the B. Mt. caused by the reactions at the abutment.

$$\text{Stress in } Q_2\,Q_1 = \frac{M_1}{p_1}$$

where M_1 is got from influence lines.

$$\text{Stress in } P_2\,P_1 = \frac{M_2}{p_2}$$

Stress in $Q_2\,P_1$ = B Mt at D of loads and reactions to left of P_2, divided by p_3.

$$= M_1 - (S_1 \times d)$$

$$\text{and } p_3 = d \cos \theta$$

$$\therefore \text{ Stress in } Q_2\,P_1 = \frac{M_1 - S_1 d}{d \cos \theta} = \left\{ \frac{M_1}{d} - S_1 \right\} \sec \theta.$$

This is suitable for the use of influence lines, the distance d being scaled or taken $= \dfrac{P_1\,P_2}{1 - p_2/p_1}$ and $\theta = \cot^{-1} \dfrac{Q_2 P_2}{P_2 P_1}$ all data being obtained from the known dimensions of the girder.

*(b) *Girder with parallel top and bottom chords. Fig. 2d.*

$$\text{Stress in } Q_2\,Q_1 = \frac{M_1}{h} \qquad \text{Stress in } Q_2\,P_1 = S_1 \sec \theta$$

$$\text{Stress in } Q_1\,P_1 = S_1,$$

where M_1 and S_1 are as before. See example (a).

(c) *Maximum Position of a series of concentrated loads at fixed distances apart (c. f. an engine.)*

(1) Bending moment at point P. Fig. 2*e*.

Ordinate at distance x from abutment

$$= \frac{x}{\lambda} (\mu)(1 - \mu). \quad \text{Where } \lambda = \text{distance of P from abutment.}$$

$$\text{Total B. Mt. } M = \Sigma\, W_r \frac{x_r}{\lambda_r} (\mu)(1 - \mu) + \Sigma\, W_l \frac{l - x_r}{\lambda_l} \mu (1 - \mu)$$

$$\text{For a max. } \frac{dM}{dx} = o$$

$$\text{and } x_r = x + \text{constants} \therefore \frac{dx_r}{dx} = 1.$$

$\therefore$ For maximum position

$$\Sigma \frac{W_r}{\lambda_r} - \Sigma \frac{W_l}{\lambda_l} = o$$

$$\therefore \frac{\Sigma W_r}{\Sigma W_l} = \frac{\lambda_r}{\lambda_l}$$

i.e. when sum of loads to right of point P bears the same ratio to sum of loads to left of point P, as the distance of P from right abutment, bears to the distance of P from the left abutment. This is the

* For further applications, graphically and otherwise, see paper by Prof. G. F. Swain, Trans. Am. Soc. C.E., Vol. 17, quoted in Johnson's "Framed Structure."

well-known relation, but the above way of deducing it is short and clear. Of course as many loads must be on the structure as will fit in, and this relation approximated to as nearly as possible with one load over the point P to be counted with the ΣW_l or ΣW_r loads as preferred.

(2) Maximum shear positive or negative. Fig. 2*b*.

It is easily seen that the maximum shear at any point occurs when the beam is loaded only from one abutment to that point.

(*d*) *Maximum reaction on cross girders.*

The question often arises in connection with cross girders, as to the maximum effects of an engine load as transferred by stringers on the cross girders. To make the case more general we will take one span on either side, the spans not being equal. Fig. 2*f*.

(Continuity is neglected; for continuous action see Cases 5, &c.)

Referring to Fig. 2*f*. Reaction at $P = \Sigma W_r \frac{x_r}{\lambda_r} + \Sigma W_l \frac{l - x_r}{\lambda_l}$

Here $x_r = x +$ constants $\therefore \frac{dx_r}{dx} = 1.$

For a maximum $\frac{dP}{dx} = 0.$

$$\therefore \frac{\Sigma W_r}{\lambda_r} = \Sigma \frac{W_l}{\lambda_l} \text{ or } \frac{\Sigma W_r}{\Sigma W_l} = \frac{\lambda_r}{\lambda_l}$$

When spans are equal $W_r = W_l$.

This result might have been observed from the similarity of Fig. 2*f* and 2*e* as regards a point P.

In practice we approximate as nearly as possible to this relation by having one load at P, and the loads to right and left have relation

$\frac{\Sigma W_r}{\Sigma W_l} = \frac{\lambda_r}{\lambda_l}$ as nearly as possible.

For equal spans the loads are placed so that the load at point P when added to ΣW_r makes the sum $\geqq \Sigma W_l$. Here ΣW_r is assumed $< \Sigma W_l$, but W_r and W_l may be interchanged in this rule.

Take the case of loco. loads. Figs. 2*g* and 2*h*.
It will be seen in Fig. 2*h* (which is the maximum position).

Loads to right $= (20 + 11) = 31$ units.
Loads to left $= (20 + 20) = 40$ units.

Adding the central load to either side makes the side to which it is added greater than the other side. This follows from the fact that at the maximum point $\frac{dM}{dx}$ changes sign.

Case 3.—BEAM SUPPORTED AT TWO POINTS, BUT WITH AN OVERHANG AT EACH END. (Fig. 3.)

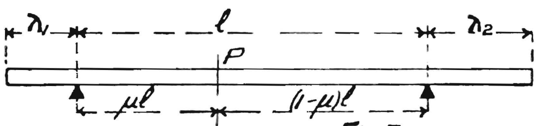


Fig. 3

As an extension of Case 2 we could combine the curves of Cases 1 and 2 to treat a beam supported at two points with overhanging ends. (Fig. 3.)

Influence line B. Mt. at point P.

Within the supports :—same as Fig. 2*a*.
Outside the supports :—Let kl = distance from support.
B. Mt. at P $= - kl\ (\mu)$ for loads on right overhang.
" " $= - kl\ (1 - \mu)$ for loads on left overhang.

The lines are shewn Fig. 3*a*.

Influence lines Shear at point P.
Within supports :—same as Fig. 2*b*.
Outside the supports:—Shear at $P = k$.

The lines are shewn Fig. 3*b*.

Cases 4—9.—CONTINUOUS GIRDERS. (Fig. 4*a*).

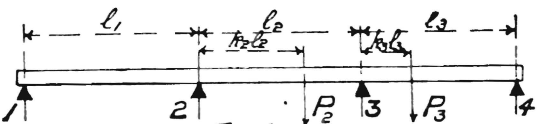


Fig. 4a

Note.—In all cases of continuous girders "k" is the distance per unit of length, from the nearest abutment on the left.

K_1 is written for $\left\{ (k) - (k^3) \right\}$ for sake of brevity.

K_2 is written for $\left\{ (1 - k) - (1 - k)^3 \right\}$ for same reason.

This is an important case as the stringers of a long bridge come under this heading, and the effect of continuity is usually allowed to give extra safety ; but the stresses occur in a different manner to ordinary beams, and as in most cases continuity is practically effected, it would seem in accordance with scientific design to know the true value of the stresses, and so have the minimum of material in accordance with the working stresses specified. In this paper the effect of yielding supports will not be investigated, as is well-known this affects the equations considerably.

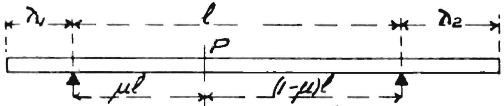
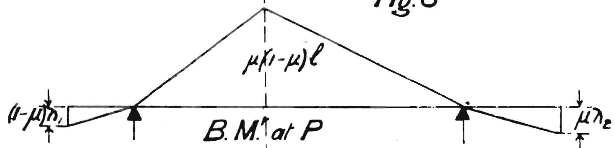


Fig. 3



B.M.t at P
Fig 3a.

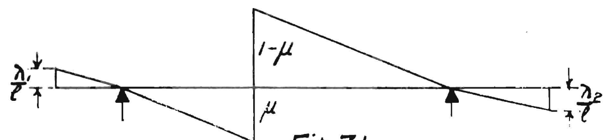


Fig 3b.
Shear at P. Reading to Left Abut:

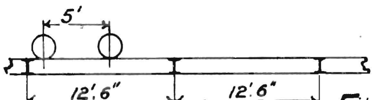


Fig. 4e.

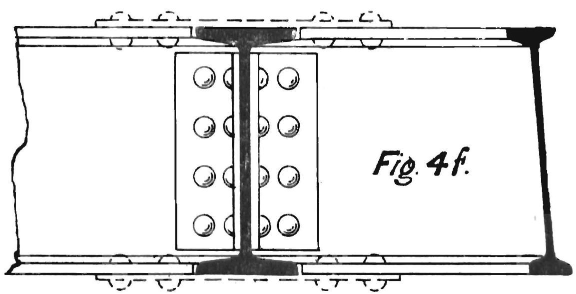


Fig. 4f.

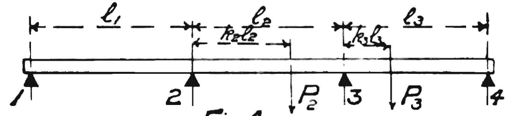


Fig. 4α

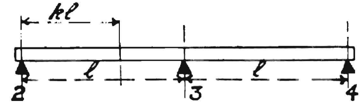
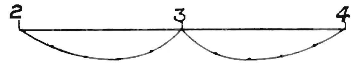
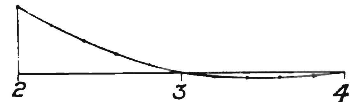


Fig 4



B.M.t at 3. Fig. 4a



Reaction at 2. Fig 4b

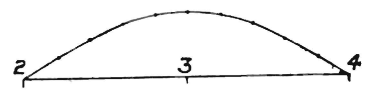


Fig 4c. Reaction at 3.

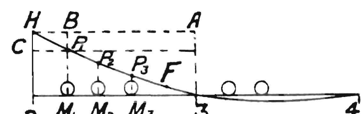


Fig 4d.

The girders are assumed continuous over three supports on the same level; and the equations as deduced below are got from the equation of three moments, proofs of which are given in many text-books. †

The equation of three moments (see Fig. 4*a*) is
$M_2\, l_2 + 2M_3\,(l_2 + l_3) + M_4\, l_3 = -P_2\, l_2^2\,(k - k^3) - P_3\, l_3^2\,(2k - 3k^2 + k^3).$
In this the last term is got from the last but one by putting $(1 - k)$ for k as is apparent from the figure, so that for our purposes we may consider only the equations in the form

$$M_2\, l_2 + 2\, M_3\,(l_2 + l_3) + M_4\, l_3 = -P_2\, l^2\, K^*$$

where l may be l_1, l_2 or l_3 according to the position of load and K may be K_1 or K_2 for the same reason.

For the purpose of getting influence lines, P_2 is to be taken unity.

The following table will be often used.

† Table of $(k - k^3)$ and $(2k - 3k^2 + k^3)$ *i.e.* K_1 and K_2.*

Read down for K_1.

k	0	1	2	3	4	5	6	7	8	9		
·0	·0000	·0100	·0200	·0300	·0399	·0499	·0598	·0697	·0795	·0893	·0990	·9
·1	·0990	·1087	·1183	·1278	·1373	·1466	·1559	·1651	·1742	·1831	·1920	·8
·2	·1920	·2007	·2094	·2178	·2262	·2344	·2424	·2503	·2580	·2656	·2730	·7
·3	·2730	·2802	·2872	·2941	·3007	·3071	·3133	·3193	·3251	·3307	·3360	·6
·4	·3360	·3411	·3459	·3505	·3548	·3589	·3627	·3662	·3694	·3724	·3750	·5
·5	·3750	·3773	·3794	·3811	·3825	·3836	.3844	·3848	·3849	·3846	·3840	·4
·6	·3840	·3830	·3817	·3800	·3779	·3754	·3725	·3692	·3656	·3615	·3570	·3
·7	·3570	·3521	·3468	·3410	·3348	·3281	·3210	·3135	·3054	·2970	·2880	·2
·8	·2880	·2786	·2686	·2582	·2473	·2359	·2239	·2115	·1985	·1850	·1710	·1
·9	·1710	·1564	·1413	·1256	·1094	·0926	·0753	·0573	·0388	·0197	·0000	·0
		9	8	7	6	5	4	3	2	1	0	k

Read up for K_2.

Case 4.—CONTINUOUS GIRDER, TWO SPANS AND THREE SUPPORTS, SPANS EQUAL TO ONE ANOTHER. (Fig. 4.)

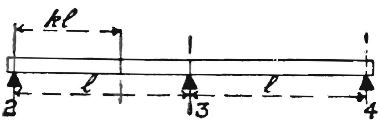


Fig 4

(*a*) Bending Moment at 3.

Since $l_1 = l_2 = l$ say.

$M_2\, l + 4\, M_3\, l + M_4\, l = -\, l^2\,(K_1)$, for load on 23.

$\therefore\ M_2 + 4\, M_3 + M_4 = -\, l\, K_1.$

$M_2 = M_4 = o.$

$\therefore\ 4\, M_3 = -\, l\, K_1$ and $M_3 = -\, \frac{1}{4}\, l\, K_1.$

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* For meaning of K_1 and K_2, see above, page 55, under "continuous girders." K may be K_1 or K_2 as required by the position of load.

† Merriman and Jacoby, "Higher Structures," part IV., p. 35.

This is a maximum when $1-3k^2=0$; *i.e.* $k=\frac{1}{\sqrt{3}}$

The figures tabulated are M_3/l.

***Table 4*a* of B. Mts. at 3.** $M_3 = -\frac{1}{4}l\,(k-k^3.)\int_{k=0}^{k=1} M_3 = -\frac{1}{16}Wl$

See Fig. 4*a*.

k	0	1	2	3	4	5	6	7	8	9
·0	·0000	·0025	·0050	·0075	·0100	·0125	·0150	·0174	·0199	·0223
·1	·0247	·0272	·0296	·0319	·0343	·0367	·0390	·0413	·0436	·0458
·2	·0480	·0502	·0524	·0545	·0566	·0586	·0606	·0626	·0645	·0664
·3	·0683	·0701	·0718	·0735	·0752	·0768	·0783	·0798	·0813	·0827
·4	·0840	·0853	·0865	·0876	·0887	·0897	·0907	·0916	·0924	·0931
·5	·0938	·0943	·0949	·0953	·0956	·0959	·0961	·0962	·0962	·0962
·6	·0960	·0958	·0954	·0950	·0945	·0939	·0931	·0923	·0914	·0904
·7	·0893	·0880	·0867	·0853	·0837	·0820	·0803	·0784	·0764	·0743
·8	·0720	·0697	·0672	·0646	·0618	·0590	·0560	·0529	·0496	·0463
·9	·0428	·0391	·0353	·0314	·0274	·0232	·0188	·0143	·0097	·0049

Maximum when $k = \frac{1}{\sqrt{3}}$. $M_3 = -\cdot0962\,l$.

When load is on 34 $M_3 = -\frac{1}{4}l\,K_2$; for arithmetical values, see Table 4*b* (β).

(b) Reactions

Take reactions upwards $\uparrow^{+}$ positive and downwards $\downarrow_{-}$ negative.

Load on span 23. Moments about 3.

$$R_2 = \frac{M_3}{l} + (1-k). \quad R_4 = \frac{M_3}{l}.$$

Moments about 4. $R_3 = \frac{1}{l}\,[-R_2 \times 2l + (2l - kl)]$

$$= -2R_2 + (2-k) = -\frac{2M_3}{l} + k.$$

Load on span 34.

$R_2 = R_4$ of above with $(1-k)$ put for k; and R_3 is found by symmetry by putting $(1-k)$ for k in above.

INFLUENCE LINE OF R_2. (See Fig. 4*b*).

Load on span 23. $R_2 = \frac{M_3}{l} + (1-k) = -\frac{1}{4}(k-k^3) + (1-k)$

$$\int_{k=0}^{k=1} R_2 = +\frac{7}{16}W$$

* See footnote on page 51.

Max. or min. when $-\frac{1}{4}(1-3k^2)-1=0$ *i.e.* $k=\pm 1$.

Load on span 34.

$$R_2 = \frac{M_3}{l} = -\tfrac{1}{4}K_2.$$

$$\int_{k=0}^{k=1} R_2 = -\tfrac{1}{16}W$$

Max. or min. when $k = 1 \pm \dfrac{1}{\sqrt{3}}$

R_4 is same as R_2 when reckoned from other end support.

Table 4b.—Reactions at 2.

(α) Load on 23. $R_2 = -\frac{1}{4}K_1 + (1-k)$. All figures positive.

k	0	1	2	3	4	5	6	7	8	9
.0	+1·0000	·9875	·9750	·9625	·9500	·9375	·9250	·9126	·9001	·8877
·1	+·8753	·8628	·8504	·8381	·8257	·8133	·8010	·7887	·7764	·7642
·2	+·7520	·7398	·7276	·7155	·7034	·6914	·6794	·6674	·6555	·6436
·3	+·6317	·6199	·6082	·5965	·5848	·5732	·5617	·5502	·5387	·5273
·4	+·5160	·5047	·4935	·4824	·4713	·4603	·4493	·4384	·4276	·4169
·5	+·4062	·3957	·3851	·3747	·3644	·3541	·3439	·3338	·3238	·3138
·6	+·3040	·2942	·2846	·2750	·2655	·2561	·2469	·2377	·2286	·2196
·7	+·2107	·2020	·1933	·1847	·1763	·1680	·1597	·1516	·1436	·1357
·8	+·1280	·1203	·1128	·1054	·0982	·0910	·0840	·0771	·0704	·0637
·9	+·0572	·0509	·0447	·0386	·0326	·0268	·0212	·0157	·0103	·0051

(β) Load on 34. $R_2 = -\frac{1}{4}K_2$. All figures negative.

k	0	1	2	3	4	5	6	7	8	9
·0	·0000	·0049	·0097	·0143	·0188	·0232	·0274	·0314	·0353	·0391
·1	·0428	·0463	·0496	·0529	·0560	·0590	·0618	·0646	·0672	·0697
·2	·0720	·0743	·0764	·0784	·0803	·0820	·0837	·0853	·0867	·0880
·3	·0893	·0904	·0914	·0923	·0931	·0939	·0945	·0950	·0954	·0958
·4	·0960	·0962	·0962	·0962	·0961	·0959	·0956	·0953	·0949	·0943
·5	·0938	·0931	·0924	·0916	·0907	·0897	·0887	·0876	·0865	·0853
·6	·0840	·0827	·0813	·0798	·0783	·0768	·0752	·0735	·0718	·0701
·7	·0683	·0664	·0645	·0626	·0606	·0586	·0566	·0545	·0524	·0502
·8	·0480	·0458	·0436	·0413	·0390	·0367	·0343	·0319	·0296	·0272
·9	·0247	·0223	·0199	·0174	·0150	·0125	·0100	·0075	·0050	·0025

Max. when $k = 1 - 1/\sqrt{3}$. $R_4 = -·0962$.

INFLUENCE LINE OF R_3. (See Fig. 4*c*).

Load on 23. $R_3 = -\dfrac{2M_3}{l} + k = \frac{1}{2}(3k - k^3)$. $\displaystyle\int_{k=0}^{k=1} R_3 = \tfrac{5}{8}W.$

Maximum when $3 = 3k^2$, *i.e.* $k = \pm 1$.

Table 4*c*.—Reactions at 3.

k	0	1	2	3	4	5	6	7	8	9
·0	·0000	·0150	·0300	·0450	·0600	·0750	·0899	·0949	·1198	·1347
·1	·1495	·1644	·1792	·1939	·2087	·2233	·2380	·2526	·2671	·2816
·2	·2960	·3104	·3247	·3389	·3531	·3672	·3812	·3952	·4090	·4228
·3	·4365	·4501	·4636	·4771	·4904	·5036	·5167	·5297	·5426	·5554
·4	·5680	·5806	·5930	·6053	·6174	·6295	·6414	·6531	·6647	·6762
·5	·6875	·6987	·7097	·7206	·7313	·7418	·7522	·7624	·7725	·7823
·6	·7920	·8015	·8109	·8200	·8290	·8377	·8463	·8546	·8628	·8708
·7	·8785	·8861	·8934	·9005	·9074	·9141	·9205	·9268	·9327	·9385
·8	·9440	·9493	·9543	·9591	·9637	·9680	·9720	·9758	·9793	·9825
·9	·9855	·9882	·9907	·9928	·9947	·9963	·9977	·9987	·9994	·9999

Load on 34.

R_3 by symmetry has same values if we measure distances from centre abutment 3, *i.e.* if we read from right end for *k* instead of from left end.

By referring to Fig. 4*a* it will be seen that the B. Mt. at 3 is a maximum when the load is at a distance ·58 *l* from the end.

Having drawn the curve to scale, by placing a diagram of the loading on the horizontal, we get the B. Mt. at 3 by multiplying the loads by the corresponding ordinates, similarly for reactions; *e. g.*, suppose the continuous span to consist of stringers each 12 ft. 6 in. long, and the live load to be a loco. with axles 5 ft. apart and loads 20,000 lbs. on each wheel. (See Fig. 4*e*).

Putting wheels over points $k = \cdot 2$ and $k = \cdot 6$, B. Mt. at 3 $=$ $(\cdot 048 + \cdot 096) \times$ 12 ft. 6 in. $\times$ 20,000 lbs. $=$ 36,000 ft. lbs. nearly.

Taking depth 15 in.

$$\text{Tension on upper rivets} = \frac{36{,}000}{1{\cdot}25} = 29{,}000 \text{ lbs.}$$

With a joint as shewn in Fig. 4*f* some such action exists. This might be provided for by putting plates at top and bottom (as dotted). The example given shews that the action is considerable. In practice such stiffness in the cross girder as would produce the figures shewn is not obtained in small bridges; but may be taken as existing in very large bridges. The figures would be increased by putting loads on the other span, and it would seem necessary to make provision for this action, as the joint cannot be made non-continuous without considerable trouble, nor in fact ought it to be, as continuity reduces the B. Mt. at the centre.

Referring to Figs. 4*b* and 4*c*, we can at once calculate the reactions at each support by putting the loading over the influence line diagram.

Having got R_2, R_3, R_4, the values of B. Mts. and shears at any point can be readily deduced.

For instance in Fig. 4*d*.

Shear at any point F is required.
Scale off P_1 M_1 &c. on 23 and 34;
take algebraical sum, this gives R_2.

The shear at F $= \Sigma\{(P_1 M_1) \times \text{load at } M_1\}$
$- \Sigma\{\text{load at } M_1 \text{ \&c.}\}$ on left of F.

Looking at Fig. 4*d* we get a geometrical representation of the effect of continuity in creating a B. Mt. at 3.

B. Mt. at 3 for a load at $M_1 = P_1 M_1 \times 23 - BM_1 \times 3 M_1$.
Where H2 represents the load to some scale.
= rectang. C3 — rectang. B3.
= rectang. CM_1 — rectang. P_1 A.

This vanishes when influence line H P_1 3 is a straight line, *i.e.* when there is no continuity.

Case 5.—THREE SUPPORTS AND TWO UNEQUAL SPANS. (Fig. 5.)

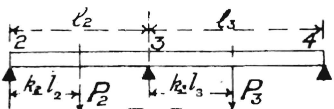

Fig 5.

Here $M_2 l_2 + 2M_3 (l_2 + l_3) + M_4 l_3 = - l_2{}^2 K_1$ for load on l_2.
$M_2 = M_4 =$ zero.
$\therefore\ 2M_3 (l_2 + l_3) = - l_2{}^2 K_1$.
Let $l_3/l_2 = r$ to keep the case general.

Bending Moment M_3. (Fig. 5*a*).

Load on l_2. $M_3 = \dfrac{-l_2}{2(1 + r)} K_1$.

$$\text{and} \int_{k=0}^{k=1} M_3 = \frac{-l_2}{8(1 + r)} W_2$$

Load on l_3. $M_3 = \dfrac{-l_3 r}{2(1 + r)} K_2$.

$$\text{and} \int_{k=0}^{k=1} M_3 = \frac{-l_3 r}{8(1 + r)} W_3$$

See Fig. 5*a*.

As an example the figures are given for $l_2 = 12$, $l_3 = 15$.
Reaction R_2 (see Fig. 5*b*).

Load on left span 23.

$$R_2 = \frac{M_3}{l_2} + (1 - k_2)$$

$$= \frac{-\,l_2}{2\,(l_2 + l_3)}\;K_1 + (1 - k_2)$$

$$= \frac{-\,1}{2\,(1 + r)}\;K_1 + (1 - k)$$

$$\int_{k=0}^{k=1} R_2 = \left\{\frac{1}{2} - \frac{1}{8\,(1+r)}\right\} W_2$$

Load on right span 34.

$$R_2 = \frac{M_3}{l_2} = \frac{-\,r^2}{2\,(1 + r)}\left\{K_2\right\}$$

$$\int_{k=0}^{k=1} R_2 = \frac{-\,r^2}{8(1+r)}\;W_3$$

The curve for this reaction is shewn in Fig. 5*b*.

Reaction R_4 (see Fig. 5*c*).

Load on left span 23.

$$R_4 = \frac{M_3}{l_3} = \frac{-\,1}{2r\,(1 + r)}\;K_1$$

$$\int_{k=0}^{k=1} R_4 = \frac{-\,1}{8r\,(1 + r)}\;W_2$$

Load on right span 34.

$$R_4 = \frac{M_3}{l_3} + k_3 = \frac{-\,r}{2\,(1 + r)}\;K_2 + k_3$$

$$\int_{k=0}^{k=1} R_4 = \left\{\frac{1}{2} - \frac{r}{8\,(1 + r)}\right\}\;W_3$$

Reaction R_3 (see Fig. 5*d*).

Load on left span.

$$R_3 = k - R_4\;(1 + r).$$

$$= k + \frac{1}{2r}\;K_1$$

$$\int_{k=0}^{k=1} R_3 = \left\{\frac{1}{2} + \frac{1}{8r}\right\} W_2$$