This has a semi-rational basis by analogy, thus,—

The deflection of a beam in terms of stress in the extreme fibre is given by the formula,*—

$$\text{Deflection} = \frac{f_b l^2}{C_l\ Ey} \quad \text{where } C_l \text{ is a constant depending on method of loading in this case} = \pi^2 \ldots\ldots\ldots \text{(i)}$$

$$= \frac{f_b}{C_2} \cdot \frac{l^2}{y}$$

In the present analysis

$$p = \frac{f}{1 + (a+e)\dfrac{y}{r^2}} \quad \text{where } (a+e) = \text{total deflection} \ldots\ldots\ldots \text{(ii)}$$

putting (a+e) from (i)

$$p = \frac{f}{1 + \dfrac{f_b}{C_2}\dfrac{l^2}{y}\dfrac{y}{r^2}} = \frac{f}{1 + \dfrac{f_b}{C_2}\left(\dfrac{l}{r}\right)^2} \ldots\ldots\ldots \text{(iii)}$$

$$= \frac{f}{1 + C\left(\dfrac{l}{r}\right)^2} \ldots\ldots\ldots \text{(iv)}$$

This is the familiar Rankine formulæ. To examine the assumptions made:—

If the deflection $= \dfrac{f_b l^2}{C_l E y}$ then in the case of a column

$$f_b = \frac{p\ (a+e)}{r^2} \cdot \frac{l^2}{(a+e)}\ \frac{f_b}{C_l E}$$

$$\therefore\ p = \frac{C_l E}{\left(\dfrac{l}{r}\right)^2} \ldots\ldots\ldots \text{(v)}$$

---

*c.f. "Equating Internal Work and External Work" for "centrally" loaded beam.

$$\text{Internal work} = 2 \times \frac{I}{2EI}\int_0^{\frac{1}{2}} M^2 dx = \text{after substituting } \frac{I}{6Ey^2} f^2 l \ldots\ldots\ldots \text{(i)}$$

$$\text{External work} = \frac{1}{2} W\ \beta = \frac{1}{2}\ \frac{4If}{yl}\beta \ldots\ldots\ldots \text{(ii)}$$

$\therefore$ Equating (i) and (ii) $\beta = \dfrac{fl^2}{12Ey}$ for 'centrally' loaded beam. Under other method of loading the constant would be changed to C_l.

It will be remembered from the previous reasoning that $p = \dfrac{\pi^2 E}{\left(\dfrac{l'}{r}\right)^2}$

where l' is the "virtual length" and the Rankine formulæ is approximately true to the extent that the virtual length may be written for the actual length and f and f_b as constants.

Alternatively

$$\text{Since } f = p + f_b \quad \therefore \quad p = \frac{f}{1 + \dfrac{f_b}{p}} = \frac{f}{1 + \dfrac{f_b}{q}\,\dfrac{q}{p}}$$

$$= \frac{f}{1 + \dfrac{f_b}{\pi^2 E}\,\dfrac{q}{p}\left(\dfrac{l}{r}\right)^2} = \frac{f}{1 + \dfrac{f_b}{\pi^2 E}\left(\dfrac{l'}{r}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots(\text{vi})$$

where l' as before is the virtual length.

Comparing (iii) and (iv) with (vi) we see (*a*) that l' is written for l;

(*b*) that $\dfrac{f_b}{C_2}$ is written as a constant.

It will thus be seen that the formula like that of Lilly (see below) confuses p and q and f and f_b.

For reading breaking test result where l may approximate to l' there may be an approach to accuracy, but, since deflection does not vary as the load nor the stress directly as the load (see Appendix C), for the purpose of deducing working stresses for design there must be a very artificial choice of constants. so much so that a simpler formula may be just as accurate.*

(8) *Gordon Formula.* The older form of Rankine has the least dimension 'd' instead of the radius of gyration, in the denominator; for the same type of section 'r' may vary as 'd' (least d mension) roughly: the formula can only be used for a very small range.

(9) *The Straight Line Formulae* are merely straight line approximations to the curve $\dfrac{l}{r}$ plotted against an empirical 'p' for centrally loaded columns; used with discretion and especially in the modern form of varying the constants according to $\dfrac{l}{r}$ they are easy of application, and when used within the range and conditions specified may be satisfactory.*

(10) *The Johnson Parabolic Formula.* Like the Straight Line, which has replaced it, is a rough approximation to the Rankine, than which it is a little easier to use.

(11) *Lilly*† deduces a modified Rankine formula. No mention is made of any essential eccentricity, thus there appears a confusion of 'p' and 'q' and of f_b and 'f.'

---

* That is, if the Statements of Appendix C., especially that under secondly, are kept in mind. See also under "More Column Design Curves," Table vi, and Tables and Curves of Sheet No. 7.

†Designs of Columns and Struts, W. E. Lilly, M.A., M.E., D.Sc., Chapman and Hall, 1908.

Let δ be the deflection (this is (a+e) or 'a' or 'e' of the present notation according to circumstance: the remainder will be put into the notation of the present paper).

The equation numbers are those of the book.

Dr. Lilly deduces $\frac{f_b}{p} = \delta \frac{y}{r^2}$ ....(3)

and $\delta = \frac{f_b}{q} \frac{r^2}{y}$ ....(8)

substituting from (8) in (3)

he gets

$$p = \frac{P}{A} = \frac{f}{1 + \frac{f_b}{q}} \qquad \text{.......................(9)}$$

putting $f_b = f$

$$p = \frac{P}{A} = \frac{f}{1 + \frac{f}{q}} \qquad \text{.......................(10)}$$

It will be seen that (3) and (8) each make

$$\delta = \frac{f_b}{p} \frac{r^2}{y} = \frac{f_b}{q} \frac{r^2}{y} \quad \textit{i.e.,}\ p = q$$

which is what (9) represents, since $p = \frac{f}{1 + \frac{f_b}{p}}$ is an identity; evidently

Dr. Lilly intended the δ of (3) to be (a+e), then the Fidler amended formula* would result.

Dr. Lilly then deduces a variation

$$\text{since } \frac{f_b}{p} = \frac{f_b}{q} \therefore \frac{p}{f_b} + 1 = 1 + \frac{q}{f_b}$$

$$\therefore \frac{p + f_b}{f_b} = \frac{f}{f_b} = 1 + \frac{q}{f_b} \quad \therefore \quad f_b = \frac{f}{1 + \frac{q}{f_b}} \qquad \text{..............(10a)†}$$

*See 2 (a) of present paper.

†Not numbered in actual text.

Now substitute the value of f_b from (10a) in equation (9)

$$p = \frac{f}{1+\dfrac{f}{1+\dfrac{q}{f_b}}\cdot\dfrac{1}{q}} = \frac{f}{1+\dfrac{f}{q}\cdot\dfrac{1}{1+\dfrac{q}{f_b}}} \quad \text{................(11)}$$

"putting $f_b = f$ so that $p = 0$."

$$p=\frac{P}{A} = \frac{f}{1+\dfrac{f}{q}\left(\dfrac{1}{1+\dfrac{q}{f}}\right)} \quad \text{...........................(13)}$$

Dr. Lilly elaborates this in the appendix putting $\omega^* = \dfrac{q}{f}$

$$\frac{p}{f_b} = \frac{\omega\,(\omega^{n-1}-1)}{\omega-1} \quad \text{.................................(24)}$$

The assumptions and substitutions seem to be somewhat sweeping.

Ideal Column Analyses.

(12) *Chapman*† gives the formula for an ideal column with the constant E (modulus of elasticity) replaced by the $\dfrac{df}{ds}$ the rate of change of stresses with respect to the strain. This refinement will only apply beyond the elastic limit, and for reading breaking test results.

(13) *Hutt*‡ assumes a neutral axis bent in a sine curve and thus the measure of the curvature $\dfrac{1}{R_1} - \dfrac{1}{R_2}$ instead of $\dfrac{1}{R}$ and obtains a formula corresponding to his assumption for a central load. Mr. Hutt has pointed out the necessity of 'e' varying with the dimensions and length of the column and the formula suggested on the diagram followed somewhat on the lines of his remarks. As it is the bending moment at the centre that is the main factor of failure, the author considers that the present treatment of the 'intrinsic eccentricity' seems to be quite as correct as any more elaborate assumption and is more useful in deductions to be made therefrom.

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*Called "a" in the book.

†P.I.C.E., 1912.

‡"Engineering," 1914.

(14) *Burgess** has solved the exact differential equation

$$\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \frac{M}{EI}$$ and proves that the correct curve of deflection differs materially from the cosine curve which neglects $\left(\frac{dy}{dx}\right)^2$, also that each load causes, or will withstand, a definite deflection which is greater than that given by the Cosine curve, though the length of the curve is less than the Cosine curve.

The author interprets Mr. Burgess' results as follows:—

His figures and results are for central loads greater than Q, consequently for columns of sufficient length to be able to withstand such loads the central breaking load may be deduced exactly.

The following figures are calculated from his table:—

H=Central load on column.
λ=Curved length of column.
l=Length of straight column.

Load. $\frac{H}{Q}$	Deflection Ratio. $\times\left(\frac{l}{\lambda}\right)^2$
1.0002	.03999
1.0020	.1263
1.0201	.3915
1.2100	1.0340

This shows that when the load is near Q the deflection increases very fast as compared with the value of the increase in load.

As an increase of .2 % in the load causes an increase in the deflection of 300%, it would seem that assuming Q as the limiting value of the load is justified, since if Q is infinitesimally increased the deflection increases enough to probably cause failure in any column of engineering proportions.

For loads less than Q on a column there can only be bending when there is some eccentricity of loading, consequently the virtual length must be for the minimum load which will keep it bent, which is the Q of the Column. The figures show that the deflection for an increase of .2% in Q is approximately $\frac{1}{8}$ length: this is beyond any probable value for (a+e) in Engineering Practice.

So far as the primary approximate equation

$\frac{d^2y}{dx^2} = \frac{M}{EI}$ is true (which it is, to well over any deflections allowable in Engineering Practice), the deductions of the present paper are justified.

The analysis of Burgess clarifies the various readings of Euler's value†, we see that Q is the least value that will bend the ideal column, that each load greater than Q gives a definite deflection, so that Q is, strictly speaking, neither a collapsing load (except from failure of the material) nor the only load that will keep the ideal column bent.

*Physical Review, March, 1917. The author's attention has been drawn to this very important paper while the present paper was in proof form by Prof. Chapman, of Adelaide.

†An exact analysis for eccentric loads has been deduced by Prof. Chapman, and sent to the author, who trusts that it will appear in the correspondence.

MORE COLUMN DESIGN CURVES.

Design of Columns. All experimental work appears to show that 'e'* varies with the length and dimensions.

Consequently it is considered better to assume 'e' with a suitable factor of safety and by trial (using a straight line formula or other empirical formula† to give a first trial) to arrive at the load that would cause the allowable stress in the extreme fibre. Using curve 3 of Sheet No. 1 would enable this to be done quickly and with an accuracy well within the limits of the assumptions that must be made.

A curve of p against $\frac{l}{r}$ could be plotted as has been done for various formulæ, but as such a curve necessarily assumes a constant $\frac{ey}{r^2}$ or φ (*i.e.*, a separate curve is required for each value of φ), a mean result can be only roughly approximate, probably no better than that given by a straight line formula. However, as the analyses of Fidler and Moncrieff state φ to vary from .15 to .6, tables and curves have been prepared for values of φ rising by .05 between the limits mentioned‡.

||Table III. and curves of Sheet No. 3 show 'f' in terms of 'p' for various values of $\frac{p}{q}$ and $\frac{ey}{r^2}$

||Table IV. and curves of Sheet No. 4 show 'p' in terms of 'f' for various values of $\frac{p}{q}$ and $\frac{ey}{r^2}$

||Table V. and curves of Sheet No. 5 show $\frac{f}{q}$ against $\frac{p}{q}$ *i.e.*, variations of stress as the load varies.

Table IV. (Sheet No. 4), gives at once the unit stress allowable for an assumed safe fibre stress, and is thus perhaps the more suitable for designing purposes.

A study of these tables and curves show factors of safety under various assumptions.

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*With a horizontal member there is an eccentricity due to the deflection due to the weight of the member. Advantage could be taken of this by making the connection eccentric on the same side of the deflection thus tending to elminate the effect of the eccentricity.—(Fidler.)

†The quadratic endorsed on the diagram is for all practical purposes exact. Graphical solutions of the quadratic are easily evolved.

‡Further experiment will probably give other values for φ, as apparently those mentioned were deduced from incorrect formulæ.

||Printed opposite the curves at the back of the paper.

To investigate effects of changes in various terms (which may be desirable to guide one's judgment) the quadratic form* may be used, but its solution may be cumbersome; the curve 3 of Sheet No. 1 is for this purpose well within the accuracy of the rest of the investigation.†

From the primary tables and curves shown, various other curves and relations may be deduced: some examples will be given of this.

(*a*) Table 3 (Sheet No. 3). Up to $\frac{p}{q} = \frac{1}{2}$, which is perhaps the limit ot safety, the curves may be approximated by the straight line (within 4% error).

$$\frac{f}{p} = (1+\varphi) + 2.5\varphi . \frac{p}{q} \qquad \ldots\ldots\ldots\ldots (1)$$

If $\varphi = \cdot 4$ and $\frac{p}{q} = \frac{1}{2}$ f roughly $= \cdot 95q = 1 \cdot 9p$

if f=16,000lbs. per sq. in. then $\frac{l}{r} = 135$

This is the condition of maximum economy for the constants taken, as the stress in the extreme fibre is the limit and the load is the safe limiting proportion of the Q of the column.

(*b*) A typical straight line formula is here tabulated on the basis of the present deductions; the results are instructive and suggestive.

The straight line formula shows φ to vary very roughly as $\frac{l}{r}$ if 'f' is assumed constant, but the stress to vary greatly if φ is a constant; a constant φ for all columns is not justifiable, and as the value of φ affects results considerably, to assume it constant leads to considerable error.

TABLE VI.—RESULTS FOR A TYPICAL STRAIGHT-LINE FORMULA.

$\frac{l}{r}$	$16{,}000-60\frac{l}{r}$	q	$\frac{p}{q}$	If $\frac{ey}{r^2} = \varphi = \cdot 4$		If f = 16,000	
				$\frac{f}{p}$	f=	$\frac{f}{p}$	φ =
60	12,400	83,000	.17	1.50	18,600	1.29	.25
80	11,200	46,000	.24	1.56	17,600	1.42	.31
100	10,000	30,000	.33	1.65	16,500	1.60	.37
120	8,800	20,000	.44	1.77	15,600	1.81	.40
140	7,600	16,000	.47	1.85	14,100	2.11	.52
160	6,400	12,000	.53	1.98	12,600	2.50	.66
180	5,200	10,000	.52	1.95	10,200	3.07	.73
200	4,000	8,000	.50	1.90	7,600	4.00	1.17
270	zero	4,100	—				

*See P.I.C.E., 1916 for slide rule solution. In most cases a graphical solution would be applicable.

†The original has $\frac{P}{Q} = \cdot 05$ represented by 1 inch.

Curves for Breaking Test Analysis and for Design.

(1). Perhaps the most useful for comparative purposes (since nearly all column literature is based on $\frac{l}{r}$) will be $\frac{p}{f}$ against $\frac{l}{r}$ for various values of φ, which will show the approximations of the straight-line formula, see Sheet No. 7, No. 1.

This gives the proportion that 'p' shall bear to 'f' for a stress 'f' to occur in the extreme fibre when $\frac{l}{r}$ is as shown by the abscissa.

The scales are the natural scales for $\frac{p}{f}$ and for $\sqrt{\frac{f}{\pi^2 E}}\ \frac{l}{r}$ *i.e.*, ordinates shall be multiplied by 'f' to get 'p' and abscissae shall be multiplied by $\sqrt{\frac{\pi^2 E}{f}}$ to get $\frac{l}{r}$, the figures for f=16,000 and f = 64,000 are endorsed.

For instance, if f=16,000lbs. and $E=30\times10^6$ ordinates shall be multiplied by 16,000 and abscissae shall be multiplied by $\sqrt{\frac{\pi^2\times30\times10^6}{16\times10^3}}$ *i.e.*, 137.

If 'f' is the modulus of rupture; to examine breaking results, say 64,000lbs. then ordinate shall be multiplied by 64,000, abscissae shall be multiplied by 68.

The last result shows the difficulty of deducing a formula for working stresses from breaking test results unless the relations just mentioned are kept in mind.

It will be seen that the curves of $\frac{p}{f}$ may be fairly approximated by straight lines, especially between ·5 and 1·2 which on the basis of f=16,000 is $\frac{l}{r}$ 75 to 165, but these presume φ constant and no mean straight line can be assumed which approximates closely to the conditions for all values of φ.

Table of straight line approximations—

$$\frac{p}{f} = \left\{k_1 - k_2\sqrt{\frac{f}{\pi^2 E}}\ \frac{l}{r}\right\}$$

φ	k_1	k_2
.2	1.06	.40
.4	.88	.33
.6	.725	.28

The straight line $p=f\left\{\cdot9-\frac{1}{3}\sqrt{\frac{f}{\pi^2 E}}\ \frac{l}{r}\right\}$ might be taken as a mean for ordinary rough practical use for mild steel

for f=16,000lbs. per sq. in. it becomes say $p=\left\{14{,}500-40\frac{l}{r}\right\}$

(2). Again φ varies with 'r' being $\frac{ey}{r^2}$ so that a constant φ cannot be assumed for various values of $\frac{l}{r}$. The effect of variations of φ for different values of $\frac{p}{q}$ is shown by Sheet No. 7, No. 2.

(3). The curve of $\frac{f}{p}$ against $\frac{f}{q}$ (see Sheet No. 7. Curves No. 3).

$$\frac{f}{p} \text{ against } \frac{f}{\pi^2 E}\left(\frac{l}{r}\right)^2 \text{ which}$$

becomes to an appropriate scale

$$\frac{f}{p} \text{ against } \left(\frac{l}{r}\right)^2 \text{ for constant 'f'}$$

See Sheet No. 7, No. 3, and 3A.

These curves are the most suitable for examining the assumption of the Rankine formula which states,—

$$f = p\left\{1 + c\left(\frac{l}{r}\right)^2\right\} \text{ i.e., that the curves are straight lines.}$$

The same feature exists as is mentioned above, under 1, that is, however good approximation these may be for breaking stresses, for design purposes where a variation in φ affects the results greatly they can only be very roughly approximate.

Curves for Direct Practical Design.

The usual problem is for certain length 'l', load 'P' and working stress 'f,' what are the dimensions required to satisfy the conditions ?

(1) A curve of $\frac{f}{p}$ against $\sqrt{\frac{f}{q}}$, become to appropriate scales

$$\left(\frac{f}{P}\right)A \quad ,, \quad \sqrt{\frac{f}{\pi^2 E}} \cdot \frac{l}{r}$$

i.e. $\quad A \quad ,, \quad \frac{l}{r} \quad$ See Sheet No. 8, No. 1.

(2) A curve of $\frac{f}{p}$ against $\sqrt{\frac{q}{f}}$ becomes to appropriate scale.

$$\left(\frac{f}{P}\right)A \quad ,, \quad \sqrt{\frac{\pi 2E}{fl^2}}\, r$$

i.e. $\quad A \quad ,, \quad r \quad$ See Sheet No. 8, No 2, and No. 2A.

If there were a curve connecting A and r (*e.g.*, with a solid circle, $A = kr^2$, with a thin circle $A = 2\pi r\, k$ for constant thickness), the exact point for A could be deduced, but as there is no definite connection between A and r, for ordinary sections, only trial and error methods are available.

These curves may be found the most suitable for design purposes in many cases for those who are not concerned with the primary formulae. The curves shown are plotted to scales for—

$$\frac{f}{P} = \text{unity and } \frac{\pi^2 E}{f l^2} = \text{unity.}$$

for instance if $f = 16{,}000$lbs. per sq. inch.

$P = 96{,}000$lbs.

$l = 10$ft.

$$\frac{f}{P} = \frac{1}{6}$$

$$\text{and} \sqrt{\frac{\pi^2 E}{f l^2}} = \frac{\sqrt{30 \times 10^7}}{16 \times (12)^2 \times 10^5} = 1.14$$

Then ordinates are to be multiplied by 6

abscissae are to be multiplied by $\frac{1}{1 \cdot 14}$ say ·88.

The designer from such a curve knows his area and so distributes the material that its radius of gyration is 'r' or vice-versa.

Conclusion.—The fact is driven home that there is no royal road to rational column design; as stated at the beginning of this paper, 'e' must be assumed (experiment will probably give formulae for its deduction), and then φ, and with the curves shewn as guides, trial and error methods will (very quickly with experience) give to any approximation desired the working stress that will cause the allowable maximum stresses, or the area and radius of gyration required.

The author considers:—

1. That test results should be studied on the basis of deducing the intrinsic eccentricity 'e,' either by plotting breaking loads against $\frac{l}{r}$ and using curves of Sheet No. 7 with 'f' as a modulus of rupture or by use of the other curves and tables such as have here given above.

2. Nearly all testing should be carried out within the elastic limit; stresses, slopes and deflections thus deduced are what should give the basis of design.*

3. Tables of $\frac{y}{r^2}$ for the various types of columns should be calculated, and thus $\varphi = \frac{ey}{r^2}$ for various lengths and values of 'e'.

The author wishes to thank students who checked the tables and prepared diagrams, and Mr. A. R. Munro, A.M.I. Mech. E., Senior Demons. trator, who supervised the work and prepared some of the diagrams: also Mr. C. N. Ross, M.Sc., B.C.E., whose strenuous discussions contributed greatly to whatever merit the paper may possess.

*Experimental work done on these lines in the laboratories of the University of Queensland confirms the theory very closely. See Sheet No. 6.

APPENDIX A.

DERIVATION OF THE EULER VALUE OF A COLUMN.

The following deductions may be interesting as showing the unique properties of Euler's value for a column*.

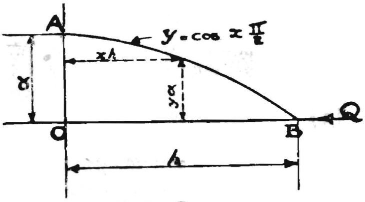

FIG. 2.

Assuming that the curve of deflection which is also the curve of Bending Moments, is a cosine curve (this is proved in most Mathematical and Engineering Text Books), we have (referring to Fig. 2), using coefficients the equation of the curve is $y = \cos x \frac{\pi}{2}$

(1) Using the $\frac{1}{EI} \iint M dx\, dx$ formula.

Deflection at B referred to tangent at $A = \frac{1}{EI} \iint M dx\, dx$.

(*a*) $\text{Slope} = \frac{1}{EI} \int M dx = \frac{Qah}{EI} \int_{o}^{1} \cos x \frac{\pi}{2} dx.$

$$= \frac{Qah}{EI} \frac{2}{\pi} \left[\sin x \frac{\pi}{2}\right]_{o}^{1} = \frac{Qah}{EI} \frac{2}{\pi} \quad \dots\dots\dots (1)$$

(*b*) $\text{Deflection} = \frac{1}{EI} \iint M dx dx = \int_{o}^{1} \frac{Qa\,h}{EI} \frac{2}{\pi} \sin x \frac{\pi}{2} d\,(xh)$

$$= \frac{Qah^2}{EI} \frac{4}{\pi^2} \left[-\cos x \frac{\pi}{2}\right]_{o}^{1}$$

$$= \frac{Qah^2}{EI} \frac{4}{\pi^2} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

But deflection $= a \therefore a = \frac{Qah^2}{EI} \frac{4}{\pi^2}$

$\therefore$ eliminating a

$$Q = \frac{\pi^2 EI}{4h^2} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

and if 2h be put $= l$ for pin ends.

$$Q = \frac{\pi^2 EI}{l^2} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3a)$$

The elimination of a shows that Q can have but one value, viz., as given by (3) or (3a), and is independent of a.

---

*This will be true only to the extent that $\left(\frac{dy}{dx}\right)^2$ may be neglected; for deflections allowable in Engineering Practice, this is justified, similarly it is neglected in deflection and continuous girder computations.

(II.) Using the $\frac{1}{EI}\int Mxdx$ formula,

Deflection at B referred to the tangent at A

$$= \frac{Q}{EI}\,ah^2 \int_0^1 \cos x \frac{\pi}{2}(1-x)dx$$

$$= \frac{Q}{EI}\,ah^2 \times F \text{ say}$$

$$F = \frac{2}{\pi}\left[(1-x) \sin x \frac{\pi}{2} + \int \sin x \frac{\pi}{2} dx\right]_0^1$$

$$= \frac{2}{\pi}\left[(1-x) \sin x \frac{\pi}{2} - \frac{2}{\pi} \cos x \frac{\pi}{2}\right]_0^1$$

$$= \frac{4}{\pi^2}$$

$\therefore$ Deflection $= \frac{Q}{EI}\,ah^2 \frac{4}{\pi^2}$ as in (2) above.

The remainder follows as above under I.*

Note—Appendices B. and C. are copies of papers submitted to the Institution of Civil Engineers † whose permission has been requested to reprint.

Under present conditions of transport and communication, it was thought that this would be the only way to present the subject at once completely and shortly.

---

*These calculations are useful as showing area of sine curve$=\frac{2}{\pi}al$, and distance of centre of gravity from B$=\frac{2}{\pi}l = \cdot 673\,l$—for a parabola the area is $\frac{2}{3}al$ and distance of centre of gravity$=\frac{5}{8}l = \cdot 625\,l$.

For portion of a sine curve the corresponding figures could be deduced, this would show the error in Moncrieff's analysis.

† Paper No. 4207, and Paper No. 4236; some slight amendments have been made.

APPENDIX B.

A PRACTICAL COLUMN DIAGRAM WITH PROOF.

The author in this paper proposes to give an analysis of the stresses in a column and to submit a diagram, using the results deduced, that will provide the designer and investigator with a simple and accurate method of knowing what happens when a column is loaded

Some experimental results will also be given* to show how the results as measured agree with those expected : it will be seen that the agreement in those quoted is close, and further investigations are in progress to verify the theoretical deductions.

The notation used is endorsed on the diagram, and will be explained also in detail in the text.

Column formulae have been deduced by theory and experiment in great number, but any column investigation must be based on the mathematical result attributed to Euler, which states that when a freely supported column has been bent, and is kept bent by means of a centrally applied load at each end,—

If Q be the centrally applied load—

l be the length of the column (pin jointed at each end)
E be the Young's modulus of elasticity for the material.
I be the Moment of Inertia of the cross section.
A be the Area of the cross section.
r be the Radius of gyration of the cross section.

$$q = \frac{Q}{A}$$

The column bends in a Cosine curve and

$$Q = \frac{\pi^2 EI}{l^2} \text{ or } q = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This result is mathematically accurate for the conditions assumed and understood in this way, the quantity 'q' becomes a property of the column, so that we may speak of the Q or 'q' of the column just as we would speak of A or I of the column, remembering the meaning of Q as stated above. In other words—when any column has been bent, Q is the only load or resistance axially applied that will keep it bent, anything more will cause collapse, anything less will allow the column to straighten. Again the bending must follow a complete cosine curve, the latter following from the facts that the double integration or differentiation of the cosine of an angle is again the cosine with the negative sign, and

if 'm' is the deflection under a central load Q
x is distance from origin taken at point of maximum deflection.

$$\frac{d^2m}{dx} = - mQ$$

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*See sheet No. 6.

The full analysis appears in text books of Mathematics and Engineering in various forms, and there is no need to repeat it, the author here has endeavoured to emphasise only its actual meaning.

However,

(1) The condition assumed cannot be realised in practice but a close approach only, which is shown by the lower portion of the diagram Fig. 2 Sheet No. 1 where 'p' approximates to 'q.'

(2) There is an 'intrinsic eccentricity' which may be either infinitesimal or appreciable, otherwise there would never be bending, but even if infinitesimal, it is sufficient to cause some bending. The 'intrinsic eccentricity' has been discussed in many papers and text books*, being due even in the most careful work to variations of E in the material of the column, slight errors in workmanship, or similar causes.

Given these fundamental facts we may proceed as follows,—

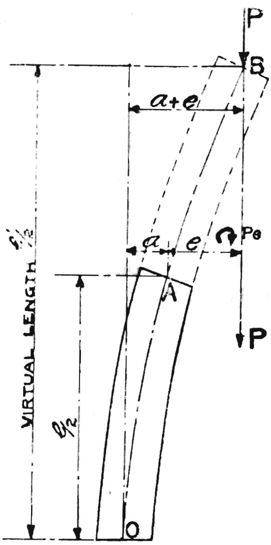

FIG. 3

Referring to the Diagram, Fig. 3.

Taking the half column (or what is the same thing considering the column as fixed at O and free at A.)

Each load as applied causes a certain definite deflection.

Let e be the eccentricity of loading (including the intrinsic eccentricity).

P be the amount of load on column

A be the area of the cross section.

$$p = \frac{P}{A}$$

$\frac{l}{2}$ be the length of the column fixed at one end.

Then at the top of the column there is a Bending Moment $P \times e$. This Bending Moment may be applied in any manner (so far as the column is concerned), that will cause a B. Mt. $P \times e$ at A. Assume it applied by lengthening the column to virtual length of $\frac{l'}{2}$ as shown in the figure, with P as a central load at B.

Then we have a column of virtual length $\frac{l'}{2}$ under a central load P which keeps it in equilibrium, that is to say P is the Q of the virtual column.

$\therefore$ from fundamental considerations

$$P = \frac{\pi^2 EI}{(l')^2} \text{ and } Q = \frac{\pi^2 EI}{l^2} \therefore \frac{P}{Q} = \frac{l^2}{(l')^2}$$

$$\therefore l' = \sqrt{\frac{q}{p}}\,l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

*c.f. Fidler "**Practical Treatise on Bridge Construction**".