accuracy of the calibration of the tank and on the measures used, so that in comparing the results of the author's experiments, with others made with different apparatus, the error might be somewhat greater. The tanks had been calibrated by filling with weighed quantities of water, and calibrations have been checked from time to time and found very accurate. For converting gallons into cubic feet the multiplier used was ·16037; as in calibrating the tank, the gallon was defined as 10 lbs. of water at 62° F.

T.—On the chronographic record the distance corresponding to one second was 0·40 inch, and the error in reading would probably not exceed $\frac{1}{50}$th second. As the duration of an experiment varied from 500 seconds upwards, the greatest error should not exceed one in 25,000 or ·00004.

A.—The diameter of the orifices were very carefully measured on a comparator in the geodetic laboratory, measurements being made on four different diameters. These measurements on the 2 inch orifice were

2·00460 2·00460 2·00462 2·00466.

The diameter of this orifice has been taken as 2·00462; to take the error at ·00005 is therefore probably overestimating it, but allows for the uncertainty of a few degrees in the temperature of the orifice; that of the room in which they were kept for a day before being measured being 60° F. The error in the area may therefore be taken as ·0001 which is equivalent to 1 in 40,000 for the 2 in. orifice, and 1 in 10,000 for the 1 in. orifice, the actual diameter of which is taken as 1·00020 inch.

H.—It is somewhat difficult to estimate the probable error in the value of the head; there are really two sources of error, the first being in setting the adjustable pointer by means of the gauge glass, and the second being due to slight variations in the head during an experiment; this variation however was never great; as the head was kept constantly under observation and the means for regulating were so good. The error in each case would not exceed $\frac{1}{200}$ in. and if both were in the same direction, totalling $\frac{1}{100}$ in., there would be an error of 1 in 1,200 at the lowest head, producing an error of 1 in 2,400, or say, ·0004 in the co-efficient.

In the case of the 1 in. orifice at the lowest head the probable errors may therefore be summed up as follows:—

| | | |
|---|---|---|
| Q | .. | ·0002 |
| A | .. | ·0001 |
| T | .. | ·00004 |
| H | .. | ·0004 |
| Total | .. | ·00074 being the greatest probable error in the worst case. |

Two experiments were always made at each head, and if these did not show good agreement, or if there were any reason to doubt one, a third was made; the greatest difference obtained between the values of the co-efficients at the same head was ·0009 and this was a rare exception. The mean of the two observations has been taken as the probable value in each case, and it seems fair to assume that the error in any case does not exceed ·0003.

The value taken for $g$ is 32·176, as determined for Montreal by Commandant Desforges in 1893.

As the diameters of the orifices were measured at a temperature of 60° F., and as the water during the experiments had an average temperature of about 40° F., the co-efficients as given are subject to a small correction on account of the diameter during the experiments being smaller than as measured. As the temperature of the orifice would be somewhat higher than that of the water, the temperature of the room being about 60°, the correction is uncertain but in any case would not exceed two in the fourth decimal place, and has therefore not been taken into account.

The values thus obtained are given in the following Table I, and plotted in Fig. 2.

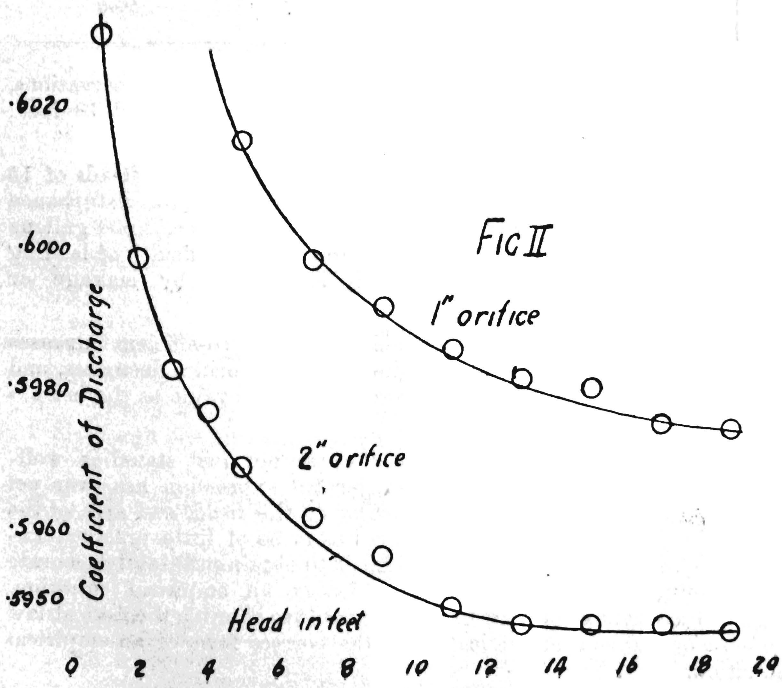

TABLE I.—VALUES OF CO-EFFICIENTS OF DISCHARGE.

| Head in Feet. | Orifice 1 in. Diameter. | | Orifice 2 in Diameter. | |
|---|---|---|---|---|
| | By Experiment. | By Formula. $C_d = \cdot 5930 + \frac{\cdot 019}{\sqrt{H}}$ | By Experiment. | By Formula. $C_d = \cdot 5920 + \frac{\cdot 011}{\sqrt{H}}$ |
| 1 | .. | ·6120 | ·6031 | ·6030 |
| 2 | .. | ·6064 | ·5999 | ·5998 |
| 3 | .. | ·6040 | ·5985 | ·5983 |
| 4 | .. | ·6025 | ·5978 | ·5975 |
| 5 | ·6016 | ·6015 | ·5970 | ·5969 |
| 7 | ·5999 | ·6002 | ·5962 | ·5962 |
| 9 | ·5993 | ·5993 | ·5957 | ·5957 |
| 11 | ·5987 | ·5987 | ·5949 | ·5953 |
| 13 | ·5985 | ·5983 | ·5948 | ·5951 |
| 15 | ·5981 | ·5979 | ·5948 | ·5949 |
| 17 | ·5976 | ·5976 | ·5948 | ·5947 |
| 19 | ·5975 | ·5974 | ·5947 | ·5946 |

Of these values, judging by the agreement of the two observations, the best are: for the 1 in. orifice those at heads of 17 and 19 feet; for 2 in. orifice, those at 1, 3, and 7 feet.

The values of the co-efficients for the 2 in. orifice at heads of 15 feet and over are a little uncertain, both on account of the disturbance which must be caused by such a large quantity of water (2,000 gallons in 12 minutes) flowing into the tank, and of the difficulty of leading this quantity into the flume without losing any by leakage or splashing.

Inspection of the curves will show that the co-efficient increases as the head decreases, and also as the size of the orifice decreases, and again that the values tend to approach a constant value as the head is increased.

That the co-efficient varies in the manner just stated is well-known, but as far as I am aware no general expression has ever yet been published giving $C_d$ as a function of the head, and area of the orifice. The variation in $C_d$ is so small as to be of little practical importance, and this also renders it difficult to obtain sufficiently accurate experimental values from which to deduce an empirical equation. Again it appears at present to be impossible to obtain a theoretical law which would give some indication of the correct form of an empirical equation.

The most complete mathematical treatment of the problem is that given by Boussinesq in an article (1) in the "Journal de Physique," for 1892, in which he gives an approximate formula for the co-efficient of discharge for given circular and rectangular orifices, and shows how the rate of discharge varies from the centre towards the periphery.

According to Boussinesq, it is impossible to determine the discharge exactly from purely theoretical considerations, but by starting from a basis of four equations, two theoretical, one experimental, and one partly both, he is able to derive an approximate formula for the co-efficient.

His theoretical equations are based, one on Torricelli's theorem giving the velocity at the surface of the jet, the other on the consideration that the stream lines at the periphery must be tangential to the plane of the orifice.

His experimental equation is based on experiments by Bazin, showing that the velocity of the central stream line in the circular orifice experimented upon was ·632, and in a long rectangular orifice without end contractions ·690, of the velocity at the free surface. His fourth equation is the well-known hydrodynamical equation (2) proving that the co-efficient of contraction can never be less than half.

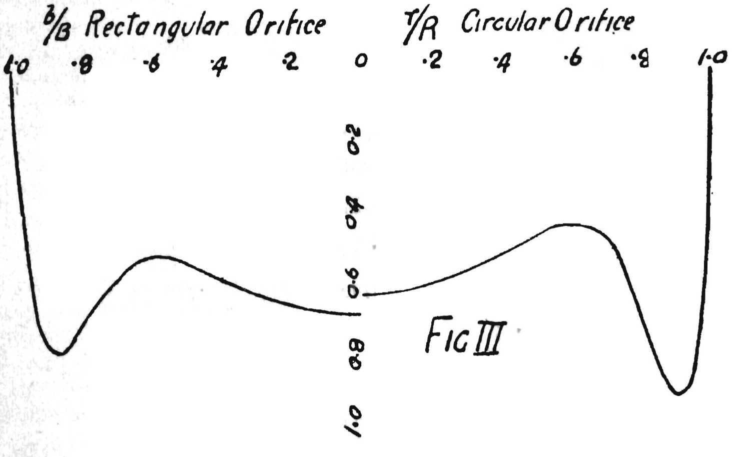


FIG III

From these four equations Boussinesq derives formulæ for the discharge per unit area, at the different points of the orifices in question as follows :—

for the circular orifice $dq = V\left(\cdot 632 + 12\cdot 2329\frac{r^{10}}{R^{10}}\right)\left(1 - \frac{r^2}{R^2}\right)$

and for the rectangular orifice $dq = V\left(\cdot 690 + 7\cdot 9272\frac{b^8}{B^8}\right)\left(1 - \frac{b^2}{B^2}\right)$

in which V in each case represents the velocity of the stream-lines on

(1). Ecoulement en Mince Paroi. Journal de Physique. Tome 1. 3me Ser. 1892.
2 Basset's Hydrodynamics, p. 29. Lamb's Motion of Fluids., p.

the free surface, as given by Torricelli's theorem, while for the circular orifice $r$ represents the distance from the centre and R the radius, and for the rectangular orifice, $b$ represents the distance from the horizontal axis, and B is one half the depth of the orifice.

The values of $dq$ obtained from the equations are given in Table II., and plotted in Fig. 3 : these curves show how erroneous is the assumption that the velocity and discharge depend only on the head, on which are based calculations for the discharge by integrating in horizontal layers, as it is clear that the velocity depends not only upon the head, but upon the position of the stream line in the orifice.

TABLE II.—VALUES OF BOUSSINESQ'S FUNCTION.

| $\frac{r}{R}$ | $(f)$ for circle. | $(f)$ for rectangle. |
|---|---|---|
| ·1 | ·6257 | ·683 |
| ·2 | ·6067 | ·662 |
| ·3 | ·5752 | ·628 |
| ·4 | ·5309 | ·584 |
| ·5 | ·4830 | ·541 |
| ·57826 | .. | ·525 min. |
| ·6 | ·4519 | ·527 |
| ·6075 | ·4516 min. | .. |
| ·7 | ·4792 | ·584 |
| ·8 | ·7005 | ·722 |
| ·85 | ·816 | ·786 |
| ·872 | .. | ·800 |
| ·9 | .. | ·777 |
| ·90205 | ·93065 max. | .. |
| ·95 | ·777 | ·589 |
| ·96 | ·688 | .. |
| ·98 | ·425 | ·297 |
| 1·0 | 0 | 0 |

It will have been noticed that the above formulæ do not take into account any variation of the co-efficients with the head or area except in so far as the experimental values of the velocity at the centre may vary with the head and area; this variation it would be useful to determine experimentally. In fact, throughout the article it is assumed that the co-efficient of contraction is synonymous with the co-efficient of discharge; which by formulæ would have the values of ·6073 and ·620, as against the experimental values of ·598 and ·626 obtained by Bazin for the orifices in question, which were 20 cent. in diameter, and 80 cent. long by 20 cent. deep respectively, under a head of about one metre. The discussion therefore although invaluable for a proper understanding of the subject of flow through orifices, is of very little guidance when dealing with the question of variation of the co-efficients.

A formula showing the variation of the co-efficient of discharge should preferably be based on formulæ giving the variations of its two components, the co-efficients of contraction and velocity. As far as I am aware, however, these two co-efficients have not been determined with accuracy over any great range of head and area, owing no doubt to the experimental difficulties, and therefore until such experiments are available, all that can be done is to form some idea of the causes of the variations and to base an empirical law on the data obtained.

It might be supposed, and Boussinesq's article seems to show, that the co-efficient of contraction is a geometrical constant, but it is suggested that the co-efficient would be affected to some extent by capillarity, the effect of which would probably be a function of the ratio of the perimeter to the area, and therefore inversely as a function of the linear dimensions of the orifice for symmetrical orifices. This theory receives some support from the fact that the value of co-efficient of discharge is altered if the orifice is at all greasy, and as will be seen later is also supported by the formulæ derived from the experimental results.

Experiments on jets of mercury, which has a very much greater surface tension than water, would throw some light on the effect of surface tension on the co-efficient of contraction.

The reduction in velocity expressed in the co-efficient of velocity is generally ascribed to viscosity which certainly must have some effect though probably a small one, but it has been suggested (3) that it is mainly due to the re-action of the outer stream lines in being deflected between the orifice and the *vena contracta*; this reaction being similar to that which occurs during the passing of a stream over a curved vane.

Whatever be the causes, a series of careful experiments is certainly needed to determine the variation of the co-efficients of contraction and of velocity. Till these are available it is impossible to derive any rational formula for the co-efficient of discharge.

When the head on an orifice is lowered to a level near the top of the orifice a free surface is formed and the conditions gradually become those for a weir; it is probable therefore that the law governing the variation of the co-efficient of discharge will cease to hold at very low heads.

Guided partly by the foregoing considerations, and by the curves, after some trial the author arrived at a fairly simple formula expressing

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(3) Strickland and Farmer—"The phenomena of jets springing from non-circular orifices." Trans. Roy. Soc., Canada, 1898-99.

the variation of $C_d$ with the head. This formula is

$$C_d = m + \frac{n}{\sqrt{h}}$$

in which $m$ and $n$ are constants for the same orifice. It is found that an expression of the above form will express the variation of the co-efficient of discharge for orifices of all sizes and shapes for which results are available, the values of $m$ and $n$ being suitably varied.

For the two orifices measured by the author the values of $m$ and $n$ which appear to give the best results are :—

for the 1 inch orifice .. $m = \cdot 5930$ $n = \cdot 019$

,, 2 ,, ,, .. $m = \cdot 5920$ $n = \cdot 011$

The values of $C_d$ corresponding to these values of $m$ and $n$, are given alongside the experimental values in Table I., and will be seen that the values given by the formulæ agree very closely with those obtained experimentally. As in most of the other experimental values to which the formula has been applied by the author the agreement is as good, there seems to be no reasonable doubt as to the correctness of the form of the expression given above.

The values of the constants cannot however be stated with such certainty as they may be varied slightly in opposite directions, and still give values of $C_d$ agreeing with the experimental values within the limits of experimental error. They would also be subject to alteration if required to express results over a greater range of head, $m$ becoming smaller and $n$ greater.

Although experiments on two orifices cannot be used to derive a formula for the variation of $C_d$ with the area, the values given above indicated that $m$ is approximately constant for circular orifices though probably larger for the smaller orifices, while it was noticed that the value of $n$ for the 2 in. orifices is about one-half of its value for the 1 in. orifice, or inversely as the diameters. It therefore seemed that with further data the formula might be extended to cover the variation of $C_d$ with the diameter.

The author therefore regrets that he did not have an opportunity to carry out, as he intended, experiments on other circular orifices which were available, as in that case the variation of $C_d$ with the head and area could have been accurately expressed. In the absence of such data it is now necessary to resort to the results given by other experimenters.

The most complete table of the values of $C_d$ for circular orifices to which the author has had access is that given on page 79 of Merriman's Treatise on Hydraulics (fourth edition), and an inspection of that table showed the value of $C_d$ for the highest heads is for every orifice except the smallest given as ·592 pointing to a value of $m$ agreeing very well with that given above. A portion of this table is reproduced in Table III., and besides the experimental values, are given values calculated from the formula $C_d = m + \frac{n}{\sqrt{h}}$