

A trial was made in some experiments at low heads, using water instead of mercury in the gauges, but there was so much oscillation in the water column that accurate readings could not be obtained, and these experiments were discontinued.

As the probable error in obtaining the loss of head was thus much greater than the errors in determining the other quantities, it is to be regretted that the pipes provided were not of greater length, though in that case there might have been some difficulty in ensuring uniformity of diameter.

For converting the readings of the mercury columns into head of water, the specific gravity of mercury has been taken at 13.6, this represents a practically constant and almost proportional error of 1 in 1000. The round figure had been taken for the purpose of quickly computing the results of the day's work at the time when the experiments were made, and later on the results did not appear to justify the labour of recalculating the values of the loss of head, using a more accurate value of the specific gravity.

It is difficult to estimate the degree of accuracy attained in reading the temperature. This was read on a standard Fahrenheit thermometer to one tenth of a degree, by holding the bulb of the thermometer in the stream of water flowing from the end of the pipe; this would differ very slightly from the temperature of the water in the pipe. Except in the runs made for the purpose of ascertaining the temperature correction, the differences of temperature were very small, and as the corrections to be applied were minute, it is thought that the error in this particular is negligible. The values of the relative fluidity were taken from G. H. Knibbs' paper, "On the Steady Flow of Water in Uniform Pipes and Channels."

The length between piezometers may be considered as known to within .01 of an inch for the straight pipe, or 1 in 7000: the accuracy of the determination of the lengths of the other pipes is no doubt much less, but cannot be stated. The weights taken for the purpose of determining the sectional area were taken to .0001 of a lb., or 1 in 3000; the mean of 2 weighings was always taken, and the probable error should not exceed 1 in 2000. The agreement in the values obtained for the diameters of the pipes indicates that the lengths and weights were determined with fair accuracy.

Consideration of the foregoing indicates that the maximum probable error should not exceed one half of one per cent., but that in most cases it should be much less. This estimate is confirmed by the agreement of individual results with the formulæ obtained for each pipe, as only a few isolated cases occur in which the departure from the straight line, drawn through the

plotted values of the logarithms of the observed results, reaches two-thirds of one per cent., while in general the agreement is much closer.

REDUCTION OF OBSERVATIONS.

The quantities plotted and used to determine the values of n and k , were the logarithms of h and w . These were taken out for each run, and unless any were obviously untrustworthy or subject of some uncertainty, the means of the values of h ., w ., and t ., were taken as representing the result of the observations at any one head. The differences were always so small that no appreciable error was introduced by taking the mean of the logarithms instead of the log of the mean. Usually three runs were made at the one head, but if there was a lack of agreement, a fourth run was taken, and in some cases a fifth. This remark refers more particularly to the earlier experiments with the straight pipe, and the first of those with the bent pipes, in which it was considered advisable to determine a number of points on the curve. When, however, it became evident that the logarithmic homologue was a straight line, not only for the straight pipe, but for the others, fewer points on the curve were taken, but a greater number of observations used to fix each point, six runs being usually made.

As it was impossible to keep the temperature of the water uniform throughout the experiments, which occupied some months, a series of runs was made on the straight pipe for the purpose of determining a temperature correction. For reducing these results, an approximate value of the index of roughness was required, and before this could be obtained, it was necessary to assume an approximate temperature correction. This was supplied by the results of Unwin's experiments on rotating disks,⁸ in accordance with which a correction was made at the rate of one per cent. for each degree Fahrenheit. With the approximate value of n thus obtained, the temperature experiments were reduced, and a more accurate value of the temperature correction derived. This was used in obtaining a more accurate value of n , and so on.

As the determination of the law of variation of the velocity with the temperature forms the subject of a later research, it is proposed to reserve the results obtained at varying temperatures for another paper.

In reducing the results for the bent pipes, the temperature correction was assumed to be the same as for the straight pipe, that is to say, the loss at the bends was assumed to be unaffected by temperature changes. This is probably correct, but, as the

tables show, the range in temperature in the experiments on any one pipe was quite small, and the corrections were well within the limits of error of the experiments.

In calculating the loss due to the bends, the results for the straight pipe were reduced to the temperature of the bent pipe in each case, so that only the ascertained temperature correction was involved. The result of this procedure is that the loss due to the bend is that which holds for the particular temperature at which the experiments were made on the bent pipe in question, and therefore, if the loss at the bends is affected by temperature, the values of λ are not strictly comparable. Seeing, however, that the loss at the bends varies as v^2 , Reynolds' formula,⁹ indicates that there would be no change with the temperature, and as the author's results are not at variance with Reynolds' formula, it is fair to assume that the results given below are independent of the temperature.

The results of the experiments are set forth in Tables I to VI., in which are given the values of h ., w ., t ., f ., and w ., corrected to the mean temperature, and the values of h . and w ., taken from these tables are plotted in Plate V. It will be noticed that the logarithmic homologue is a straight line, not only in the case of the straight pipe, as is now well known, but in the case of the bent pipes, which was rather unexpected. Consequently, it is possible to represent the results for each pipe by the expression:—

$$\dot{h} = \dot{k} + n\dot{w} \quad \text{or } h = kw^a$$

The values of \dot{k} . and n . for the various pipes, are set forth in Table VI.

Although the values of \dot{k} . and n ., given in this table, exhibit a definite progression with the values of ϕ ., they do not increase in proportion, as the pipes were not all of the same length, and the loss at the bend is, of course, greater in proportion for the shorter pipes.

The loss of head due to a bend, could, therefore, not be taken directly from the curves or tables, and it was necessary to derive it in the following manner:

From the series of experiments in the straight pipe, the loss of head per unit length, or the equivalent slope $s = \frac{h}{l}$, was found to be given by the expression

$$\dot{s} = \dot{h} - \dot{l} = \bar{2}.3677 + 1.717 v$$

at 60 deg. Fahr. The total loss of head due to friction in any of

the bent pipes, is evidently obtained by adding to the value of s , as given above, the logarithm of the length of the pipe in question, and then making corrections for the difference in temperature and diameter. As stated before, the temperature correction applied was that derived from the experiments on the straight pipe.

In making a correction for the slight differences in diameter, Osborne Reynolds' formula¹⁰ was used. According to this formula, the loss of head in a pipe varies inversely as the $(3-n)^{\text{th}}$ power of the diameter; and although doubts have been expressed as to the exactness of this formula,¹¹ it may be taken to be nearly correct; in any case, the correction to be applied would be very small, and the probable error is negligible.†

The loss of head due to one bend, is then evidently one fourth of the difference between the loss of head found experimentally for the bent pipe, and that derived as above for a straight pipe of the same length as the bent pipe.

In this way, the loss due to one bend was calculated for different velocities from one foot per second to 10 feet per second, which is about the range of velocities in the experiments. The results obtained are set forth in Tables VII to X, and are plotted on Plate VI.

Examination of these results shows that the loss does not vary exactly as the square of the velocity, and that it does not follow a simple index law. It can be more precisely represented by an expression of the form $\lambda = a v + \beta v^2$, but as the use of this form would have involved the calculation of two co-efficients and the determination of the law of their variation with the angle of the bend, it has been thought simpler to derive a co-efficient on the assumption that the loss varies exactly as the square of the velocity. For this co-efficient, there has been taken the mean value of λ/v^2 for the range of the experiments in each case.

This assumption is justified by the theory usually advanced to account for the loss of head at bends, which indicates that a proportion of the velocity head $\frac{v^2}{2g}$ is lost in shock at the bend.¹²

It will be noticed that the value of $\frac{\lambda}{v^2}$ is most constant in the case of the pipe with the greatest angle, that is to say, in which the loss of head due to the bend is greatest, and that it is least constant where the loss is smallest, and where any error

(10) Phil. Trans., Vol. 174.

(11) G. H. Knibbs, Proc. Roy. Soc., N.S.W., Vol. 31.

† Since writing the above, an account has been published of some experiments on large wooden stave pipes, the results of which confirm Osborne Reynolds' formula in this particular.

(12) See Bovey's Hydraulics, Page 92. Also Article Hydro-mechanics, Encyclopædia Britannica.

would therefore be proportionately greatest. It is possible, therefore, that the observed deviation from the theoretical law is due to some error in the experiments or in the method of reduction; and the regularity of the deviation would indicate that such errors were constant in their nature and incidence, and so applied to all the observations. In comparing the accuracy obtained in the values of $\frac{\lambda}{v^2}$ with that claimed for the experiments in each pipe, it must be borne in mind that the loss due to the bend is only a fraction of the loss due to friction, and that this loss is the difference between two observed values, both of which are subject to error; the percentage accuracy would therefore be much less, especially in the case of the pipe with right-angled bends.

The slight disagreement, noted above, with the theory that the loss of head at an elbow should vary exactly as the square of the velocity, may, however, be explained by assuming that the total loss due to the bend is made up of two parts, one the loss in shock—a purely local effect—varying as v^2 ; and the other representing an increased loss in an undetermined length of straight pipe below the bend. The local effect is the formation of an eddy at the inside of the elbow, as may easily be observed in a bent glass tube, in which it may be seen that the effect is somewhat irregular, especially if there be any air in the water. The result of this eddy is no doubt to increase the disturbance in the fluid for some distance below the bend, and so to make the loss of head in this distance somewhat greater than that in the other portions of the pipe. This secondary effect is probably only slightly affected by the value of the angle of the bend, or in any case increases at a much slower rate with the angle than does the local loss. It seems probable also that this secondary effect will vary with the degree of roughness of the pipe. If this supposition be correct, the secondary effect should be less noticeable in very rough pipes, in which the molecular disturbance is considerable even in the straight portions. This secondary effect, if it exists, is evidently of very little practical importance.

Seeing that the loss of head at the bend is always positive and must increase with the angle through the possible range from 0 deg. to 180 deg., it is clear that it must be a function of $\sin \frac{\Phi}{2}$. The mean values of $\frac{\lambda}{v^2}$ for the approximate range of the experiments in each case, are, therefore, tabulated with the values of $\sin \frac{\Phi}{2}$ in the following Table XI, and are plotted against $\sin^2 \frac{\Phi}{2}$ on Plate VII.

TABLE XI.

ϕ	$\text{Sin}^2 \frac{\phi}{2}$	$\frac{\lambda}{v^2}$	$\frac{\lambda}{v^2} / \text{Sin}^2 \frac{\phi}{2}$
90.6°	.506	.0222	.0439
119.8°	.748	.0309	.0414
135.2°	.855	.0403	.0471
150.2°	.933	.0413	.0442

Mean0441

It should here be noted that the angles in the pipes could not be measured with great accuracy, and, being derived from outside measurements, are subject to some uncertainty. The four angles in the one pipe were not exactly equal, and the mean values have been taken in the above table. This may account to some extent for the irregularity in the values given in the last column.

It will be noticed, however, that the pipes for which the agreement is reasonably close, viz., those with angles of 90deg. and 150deg., are those in which a greater number of experiments was made, and to which, therefore, greater weight should be given. It is probable, therefore, that the mean value of .044 is not far from the truth.

Very few experiments appear to have been made by other observers, with a view to determining the loss of head due to bends in pipes. Those published by Weisbach in his work, "Die experimental Hydraulik," are usually quoted in text-books on hydraulics, and it will be of interest to compare the results obtained in the two series.

Weisbach's formula is:—

$$\lambda = m \frac{v^2}{2g} \text{ where}$$

$$m = .9457 \text{ Sin}^2 \frac{\phi}{2} + 2.047 \text{ Sin}^4 \frac{\phi}{2}$$

The formula deduced above as representing the results of the experiments described in this paper is

$$\lambda = .044 \text{ Sin}^2 \frac{\phi}{2} \cdot v^2$$

$$= 2.831 \text{ Sin}^2 \frac{\phi}{2} \cdot \frac{v^2}{2g} = m \frac{v^2}{2g}$$

g being equal to 32.176 in Montreal. The corresponding values of m are set forth in the following Table XII., and plotted on Plate VIII.

TABLE XII.

Values of m in formula $\lambda = m \frac{v^2}{2g}$

ANGLE.	90°	120°	130°	135°	140°	150°
m by Weisbach's formula	.984	1.861	2.158	2.	2.431	2.664
m by Author's „	1.415	2.123	2.325	2.416	2.500	2.641
m by Experiment	1.416	1.988	—	2.593	—	2.658

It will be seen that the loss, according to Weisbach, is materially less than that found in the experiments just described, except in the case of the pipes with large angles; the results being practically identical for the 150deg. pipe. The most obvious explanation of the discrepancy is that it is due to the difference in diameters of the pipes used in the two series of experiments, the diameter of the pipes used by Weisbach being 1.2 inches.

Unfortunately, the detailed account of Weisbach's experiments is not available, but they must have been accurate to justify the values given for the constants, and to allow of the deduction of an expression containing two powers of $\sin \frac{\phi}{2}$. Weisbach's value for a 90deg. bend is supported by theory, according to which the whole of the velocity head is lost at a right-angled-bend. A series of experiments on pipes having a diameter of say $\frac{3}{4}$ in. would settle the question as to whether the loss of head at bends is influenced by the diameter of the pipe.

In explanation of the apparent incompleteness of this series of experiments, it should be stated that they were undertaken during the preparation of apparatus for a more important research, and were abandoned when this apparatus was ready for use. Pipes had been prepared, having angles of 30deg., 45deg., and 60deg., and had time been available, the variation of λ with ϕ could have been determined with greater accuracy.

A number of experiments were made on a pipe with angles of approximately 128deg., the results of which are represented by the expression

$$\begin{aligned} h &= 3.8564 + 1.867 v \\ &= 1.5147 + 1.867 v \text{ at } 66.85^\circ \text{ Fahr.} \end{aligned}$$

The length of this pipe was 68.66 inches. The results have not been included in the paper, as the values of λ proved to be so much in excess of those for the other pipes, that it was evident that there was some considerable error. Judging by its external appearance, this pipe was badly made, and it is supposed that the bends were not quite clean inside, and that owing to some

projections at the angles, the loss has been increased. The results of the experiments on this pipe were consistent with one another, and it was only recently when the values of λ came to be calculated, that it became evident that something was wrong. By that time, the pipe had been destroyed in the unfortunate fire in the Engineering School at McGill University, and it was therefore impossible to have the pipe examined with a view to ascertaining the cause of the discrepancy in the experiments on this pipe.

Since writing the foregoing, the author's attention has been drawn to some experiments carried out by Dr. Brightmore at the Royal Engineering College at Cooper's Hill, and published in the Proceedings of the Institution of Civil Engineers, Vol. clxix. These experiments were made on rusted iron pipes of 3in. and 4in. diameter, and on right-angled bends and curves. The tabulated results are not published, and the curves indicate that the experiments were not very accurate. The loss at the bends was derived, as in the experiments described in this paper, by measuring the loss in a length of pipe including the bend, and subtracting the loss in the equivalent length of straight pipe. The loss at the right-angled bend is stated to vary very nearly as v^2 and is the same for both pipes, the co-efficient m in the expression

$$\lambda = m \frac{v^2}{2g} \text{ being } 1.17 \text{ in both cases.}$$

This value, it will be noticed, is intermediate between Weisbach's and the author's, and it appears that the discrepancy cannot be due to the difference in diameters. It may be noted that the value of n for the pipes used by Brightmore was very nearly 2, showing that the pipes were quite rough. The value of n in Weisbach's experiments is not available.

TABLE I.

STRAIGHT PIPE.

h	w observed	t	f	w at 59.9° F	$h=3.1454+1.717w$ at 59.9° F
1.9081	2.1141	60.0	.1992	2.1140	
0.0547	2.1978	59.25	.1945	.1984	
.4333	.4232	61.5	.2084	.4216	
.6992	.5783	61.1	.2060	.5771	
.8026	.6359	61.1	.2063	.6346	
.8044	.6341	58.2	.1878	.6356	
0.8351	2.6527	58.7	.1909	.6539	

TABLE II.
PIPE WITH FOUR 90° BENDS.

h	w observed	t	f	w at 63.23° F
0.2637	2.1661	63.0	.2175	2.1662
.4489	.2674	63.23	.2190	.2674
.5737	.3360	63.25	.2190	.3360
.6753	.3931	63.72	.2219	.3928
.8870	.5062	61.83	.2104	.5070
.9238	.5273	62.33	.2135	.5278
0.9719	.5538	62.37	.2137	.5543
1.0202	.5788	63.32	.2196	.5787
1.0555	.5996	63.32	.2196	.5995
1.0907	2.6204	64.74	.2284	2.6195

$h = 3.6140 + 1.827 w$
at 63.23° F
 $= 1.3653 + 1.827 v$

TABLE III.
PIPE WITH FOUR 120° BENDS.

h	w observed	t	f	w at 71.0° F
0.5390	2.2816	71.12	.2662	2.2816
.8543	.4516	70.75	.2641	.4517
1.1451	2.6082	71.10	.2662	2.6082

$h = 3.7281 + 1.856 w$
at 71.0° F
 $= 1.4266 + 1.856 v$

TABLE IV.
PIPE WITH FOUR 135° BENDS.

h	w observed.	t	f	w at 72°.08° F
1.1691	2.5729	72.41	.2737	2.5728
1.1689	.5725	71.66	.2693	.5726
0.8615	.4094	71.18	.2667	.4097
0.6976	2.3229	73.08	.2776	2.3226

$h = 3.8595 + 1.885 w$
at 72.08° F.
 $= 1.4854 + 1.885 v$

TABLE V.
PIPE WITH FOUR 150° BENDS.

h	w Observed.	t	f	w at 73°.53° F
0.6703	2.2947	73.73	.2813	2.2946
.6758	.2977	73.63	.2808	.2977
.7163	.3187	73.64	.2808	.3187
0.9629	.4490	73.14	.2778	.4492
1.0563	.4976	73.52	.2801	.4976
1.1744	2.5607	73.54	.2803	2.5607

$h = 3.9021 + 1.895 w$
at 73.53° F.
 $= 1.5044 + 1.895 v$

TABLES VI.
SUMMARY OF TABLES I. TO V.

Φ	l	d	k	n	t° F.
0	72.815	.3725	3.1454	1.717	59° 9
90.6°	71.460	.3728	3.6140	1.827	63.23
119.8°	73.995	.3752	3.7281	1.856	71.0
135.2	79.435	.3752	3.8595	1.885	72.08
150.2	90.070	.3748	3.9021	1.895	73.53

TABLE VII.

Loss due to one 90° bend.

v	λ	$\frac{\lambda}{v^2}$
3	.2056	.0228
4	.3594	.0224
5	.5543	.0222
6	.7875	.0219
7	1.0605	.0216
8	1.3716	.0214
9	1.7210	.0212
10	2.1070	.0210
Mean		.0218

TABLE VIII.

Loss due to one 120° bend.

v	λ	$\frac{\lambda}{v^2}$
3	.283	.0314
4	.498	.0311
5	.771	.0309
6	1.100	.0306
7	1.485	.0303
8	1.927	.0301
9	2.418	.0299
10	2.972	.0297
Mean		.0305

TABLE IX.

Loss due to one 135° bend.

v	λ	$\frac{\lambda}{v^2}$
3	.366	.0406
4	.649	.0406
5	1.010	.0404
6	1.448	.0402
7	1.964	.0401
8	2.556	.0399
9	3.220	.0398
10	3.966	.0397
Mean		.0402

TABLE X.

Loss due to one 150° bend.

v	λ	$\frac{\lambda}{v^2}$
3	.369	.0410
4	.660	.0412
5	1.034	.0413
6	1.488	.0413
7	2.026	.0413
8	2.645	.0413
9	3.340	.0412
10	4.126	.0413
Mean		.0412