REASONING WITH FRACTIONS: SUPPORTING PRE-SERVICE TEACHERS' LEARNING

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ABSTRACT
We report on a mathematics education course for primary pre-service teachers (PSTs) in which both mathematical and pedagogical learning is essential. Part of this course is devoted to supporting PSTs in (a) strengthening conceptual understanding of fractions and (b) coming to imagine ways in which fractions can be effectively introduced in primary classrooms. In collaborative group work on fraction activities, PSTs generated solutions that were very different from those of year 3-5 pupils. This was unexpected and led to new insights and learning for the instructors. We draw implications about supporting the PSTs' development of deeper mathematical reasoning in circumstances where prior mathematical learning limits their problem-solving skills and creativity. This work contributes to a larger effort of developing resources that could enhance adults' learning across STEM disciplines.

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INTRODUCTION
Primary teachers can play a crucial role in keeping STEM learning pathways open for young children by encouraging pupils' curiosity and cultivating their interest in STEM disciplines. It is therefore highly problematic that not many primary teachers have had opportunities to experience the kinds of learning that result in deep understandings of big disciplinary ideas. In mathematics, past learning experiences of pre-service teachers (PSTs) are often comprised of memorising formulas and learning to "apply" these in repetitive exercises. Primary teacher education programs thus often aim to supply relevant discipline-specific learning experiences for PSTs, and then use these shared experiences to examine the teaching that made them possible.

Reflecting the wider phenomenon, PSTs' reasoning with fractions and proportional relationships is rarely strong (cf. Lamon 2007). We drew on the first author's expertise in supporting primary pupils' learning in this area (Cortina, Visnovska & Zuniga 2014a; Cortina & Visnovska 2015) as we organised the third-year primary mathematics education course. We planned for PSTs to experience meaningful mathematical learning in which their fraction understanding could be strengthened. Our in-depth focus on a specific mathematical idea embodied the goals of an OLT-funded project *Inspiring Mathematics and Science in Teacher Education (IMSITE)*, which aims to enhance teacher education by creating opportunities to transcend disciplinary boundaries at the university. In this paper, we document initial insights about PSTs solution approaches and how these were shaped by their prior mathematical learning. We draw implications for our future work in mathematics teacher education courses.

MATHEMATICAL LEARNING IN TEACHER EDUCATION
The mathematical goals of mathematics education courses at the university level are multiple and varied and include shaping PSTs' beliefs and conceptualisations of mathematics, as well as building their mathematical confidence and competence. Coming to view mathematics as (a) different from a pre-set collection of rules and skills (Ernest 1989), and as (b) a useful and relevant tool in personal decision making (Cooke & Walker 2015) are two of the transitions many PSTs need to make as they learn to teach mathematics for understanding, and support STEM learning more broadly. Our inclusion of the sequence of 'lessons' on fractions within the course was intended to illustrate to PSTs how mathematics can be experienced in these ways.

Ball and Bass (2003) demonstrated that the job of teaching mathematics carries mathematical demands that go beyond common content knowledge. Common knowledge of content requires that a person is able to solve a mathematical problem accurately, and many non-teachers are likely to hold such knowledge. In contrast, solving mathematical problems *of teaching* involves, for example, inspecting alternative solution methods that students present in classroom that are often unknown to the teacher. The teacher needs to examine mathematical structure and principles that underlie these methods, and judge whether or not a method is valid and can be generalized. The ability to do so effectively needs to be cultivated in mathematics education courses. While PSTs learn in and from situations that are likely to arise in primary classrooms, the mathematical content they are learning exceeds their past learning in primary mathematics. It includes envisioning different solutions to a mathematical task, producing explanations and representations with an intention of helping others to follow an argument, and judging mathematical adequacy of others' explanations. We planned for experiences of this kind to be included in the sequence of 'lessons' on fractions.

'FRACTIONS AS MEASURES' LEARNING SEQUENCE

The instructional tasks used to teach fractions in schools almost exclusively fall within what Freudenthal (1983) characterises as *fraction as fracturer* situations, where a whole (often a food item) is being cut or split into equal-sized parts. We elaborate elsewhere (Cortina, Visnovska, & Zuniga 2014b) on how these types of instructional supports result in fraction images that are counterproductive to developing a mature understanding of fractions. An example of a confusion that directly results from one such taught, counterproductive image is a novice learner's puzzlement over improper fractions, because *it is not experientially sensible* to have 7/4 of a cake. Attempts to address puzzlement involve changing the very models and definitions of what students were previously taught fractions to be ("Just imagine two cakes now, but only one of them is the 'whole'"). To many students such changes of definitions indicate—likely for the first time in their schooling—that mathematics is arbitrary, is about remembering tricks (which definition of fraction to call on in which situation), and that in doing mathematics, they cannot rely on their own judgement and logical reasoning.

Our instructional approach, developed in a series of classroom design experiments (Gravemeijer & Cobb 2006), is based on a different type of everyday situations, where fractions are used to compare sizes (e.g., lengths) of separate objects (*fraction as comparer*, Freudenthal 1983; cf. fraction as measure, Kieren 1980). Our research in primary classrooms indicates that the *Fractions as Measures* sequence is a valuable resource in supporting initial fraction learning and that it does not introduce arbitrary experiential roadblocks akin to one described above (Cortina & Visnovska 2015). It aims at cultivating an image of unit fractions as the amount of an attribute that is (a) separate from a reference unit, and (b) fulfils an iterative condition with respect to the reference unit.

We cultivated this image in a sequence of activities where the reference unit was represented by the length of a wooden stick (about 24 cm long) and the unitary fractional amounts were lengths of plastic drinking-straws. Students would judge a straw to be 1/2 as long as the stick (reference unit) when two iterations of the length of the straw covered the exact length of the stick (see Figure 1). A straw would be judged as being 1/3 as long as the reference unit when three iterations of its length covered the exact length of the stick, and so on.

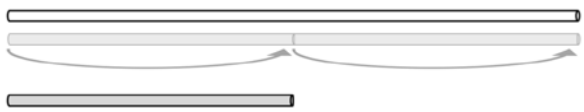

Figure 1. A half as the length of something that requires two iterations to cover the length of the reference unit.

To cultivate this image of unit fractions, we used a series of activities based on the reinvention of linear measurement. The overarching narrative of the activities involved learning about the ways in which a fictitious group of people, living in pre-colonial Mesoamerica, measured. Using this scenario, we first established with the students the need for standard unit of measurement (rather than measuring with body parts), the stick. We then brought students' attention to the fact that the length of many things would not correspond to a whole number of iterations of the stick and helped them to recognise the need for creating and using units of measurement shorter than the length of one stick. The production of straws that fulfilled specific iterative conditions was then introduced as the solution developed by the ancient people. Crucially, within this type of pedagogy, a new tool (or a definition) is introduced only *after* a strong need for a new tool (or a definition) have been established in the classroom. In this way, we cultivate students' views of mathematics as being both useful and logical.

Students were first asked to produce a straw, called *small of two*, of such a length that two iterations of it would exactly cover the length of the stick. They were given a drinking straw about 15 cm long and asked to iterate it along the stick. If the two iterations did not exactly cover the length of the stick, students were asked to manipulate the length of the straw so that it did, either by reducing its size using a pair of scissors, or by using a longer straw. By repeated trials, students homed in on the specified length.

Through engaging in activities of this kind, primary students in multiple classrooms where we worked came to first reason proficiently about inverse order relation ("the more times something fits along the stick, the smaller it has to be") and subsequently conduct comparisons of fraction lengths both smaller and larger than the stick (Cortina & Visnovska 2015). We therefore conjectured that PSTs could also experience how these insights can be developed as they engage in these activities as learners.

In these activities, we planned for the PSTs to work in pairs or small groups, to engage in realistic problem scenarios, and to subsequently reflect on how initial fraction learning can be developed in a primary classroom. We conjectured that the activities would support PSTs in adopting productive conceptualisations of mathematics, and specifically help them "endorse attitudes regarding the usefulness and need of mathematics in everyday life and the relevance of classroom mathematics to student lives" (Cooke & Walker 2015, pp. 43-44). Cooke and Walker suggest that such conceptualisations shape whether teachers make connections between mathematical ideas and integrate mathematics meaningfully.

BACKGROUND OF THE STUDY

Intensive, 9-week course devoted to mathematics pedagogy and content, attended by 104 primary and middle years PSTs, involved a weekly 2h workshop and 2h tutorial session. Most of the enrolled students were in the third year of their bachelor's degree. Fractions were explored in four consecutive whole-cohort workshops, each session lasting up to 30 min. Prior to these workshops, PSTs completed an 'initial thoughts' quiz that targeted their *mathematical knowledge for teaching* (Ball & Bass 2003), and in particular their ability to produce adequate representations of relationships that involve fractions. The main intent of the quiz was to provide justification (to both researchers and PSTs) for the need to explore fractions in greater depth. Of the PSTs who consented to using their data for the study, 62 took the quiz. Their responses to one of the seven tasks illustrate the range and distribution of fraction understandings within the cohort. Task F asked PSTs to

F. Draw a picture to explain $\frac{1}{3} \times \frac{2}{5}$

Of the 62, 17 PSTs either did not attempt the task or crossed over their attempted solution. Nine PSTs produced (correct or incorrect) calculations but included no drawing. Similar to responses on Fig. 2, 26 PSTs included a drawing where two fraction quantities were represented individually, but the drawing did not capture the meaning of multiplication. The remaining 10 PSTs' drawings adequately represented quantitative meaning of fraction multiplication.

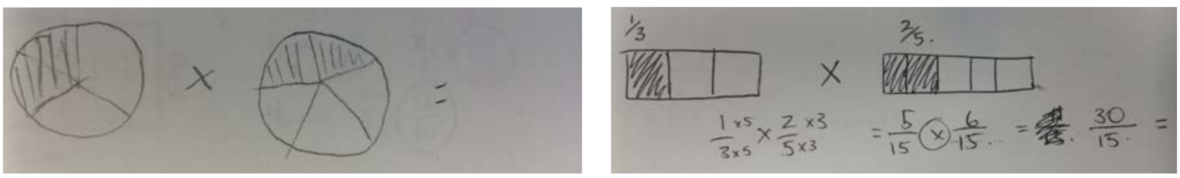

Figure 2. Two examples of typical PSTs responses to task F.

We concluded from the quiz that most of our PSTs had not yet developed a solid understanding of fractions as numbers that represent quantities (and did not have a strong sense of how multiplication works when repeated addition is not applicable). These results were consistent with mathematical insights and limitations of PSTs in this course in prior years. They were also consistent with broader research literature on challenging nature of fraction learning (Lamon 2007), and provided justification to focus our teaching efforts on initial fraction learning.

METHODOLOGY AND DATA

We conceived of the study as a classroom design experiment in a tertiary setting (cf. Gravemeijer & Cobb 2006). We were interested to find out how PSTs would engage with the sequence activities, whether and in what ways these would promote their own mathematical learning, and whether they would come to form insights about using learning sequences like the fractions one in a primary classroom. In this paper, we address the first of these questions, as our findings were surprising.

In addition to the initial quiz, collected data include videos of a small group of PSTs engaging in the activities, class audio recording, copies of PSTs' work, researchers' log of design conjectures, anonymous online student surveys, and exit interviews conducted with seven volunteers.

We aim to document unanticipated ways in which PSTs approached the initial problem-solving activities of the fractions sequence, in the first 30-min session. We transcribed the session to document PSTs' solution approaches and explanations and compared these with approaches of primary students who engaged in the fraction sequence in prior classroom design experiments. We analysed mathematical potential and limitations of models used by PSTs, and formulated conjectures about possible modifications of the sequence for more productive use with future PST cohorts.

WORKSHOP VIGNETTE AND DISCUSSION

The first fractions session followed activities related to introducing measurement of length where PSTs measured objects in class first with their body parts and then with standard unit of measure, the stick. The pedagogical notion that we aim to only propose a solution to students once they have developed an appreciation for the problem had also been discussed. In the session, we were developing a need for shorter units of measure, which would be systematic and manageable, and would be capable of accounting for a variety of possible lengths that are 'left over' when measuring with the stick. Ancient elders, the PSTs were then told, came up with a notion of *smalls* as a solution to this problem.

Each PST then attempted to produce the small of two, using the stick, pair of scissors, and a drinking straw shorter than the stick. We expected that, like primary students, PSTs would use trial and error to adjust the straw to the desired length: such that two iterations of the length of the straw would cover the length of the stick exactly. Most importantly, we assumed that given the context of the measurement, PSTs would find it reasonable to use iteration of a single straw as a means to check whether the length of this straw satisfied the condition.

Instead, they proceeded to work with straws differently. Many students, working in pairs or groups of three, joined 2 straws together, adjusted the length of the long straw to match the length of the stick, bent it in the middle and cut it to produce two short straws, each half the length of the stick.

Instructor:	What is this? (pointing to the extended straw)
PST1:	It's the extra bit at the end cause we wanted to make it long
PST2:	We put them together
Instructor:	Why?
PST2:	To make one long one
Instructor:	Yeah, but why do you want a long one?
PST1:	Because one straw is not long enough to make (points at two ends of the stick, see Figure 3)

Figure 3. PST1 indicates the length of straw that she required to make a small of two.

… …
PST1: (jumps in) You want to make it (the small of two) a perfect length.
Instructor: Yes?
PST1: Because if you- We make one long one like this, and then fold it in half to make it perfect half, and then cut.
Instructor: So there is no other way than folding to make a small of two?
PST1: Not- (If there was) I don't know if it would be perfectly half though.

After a similar discussion took place with another group, where students were similarly skeptical about existence of 'other than folding' method, the instructor gathered everyone's attention and shared that while this was the strategy in most of the PST groups, very few students in primary classrooms used it. Indeed, the straws were deliberately shorter than the stick so as to discourage bending in half, and for primary students that have worked. She also shared that by the time we reach university, we are so used to fractions as folding and cutting that preference for this strategy is understandable. She then asked PSTs to think of another way to make the small of two: How did young pupils do it?

Interestingly, even after PSTs adopted a trial and error strategy, they typically cut two straws of the same size and then aligned these next to the stick. In this way, instead of comparing relative lengths of 'estimated 1/2' and the unit, they compared lengths of two units: one created by two estimated halves, and the actual unit whole (see Fig. 4a). In contrast, primary pupils genuinely adopted measurement metaphor and used only one straw, which they iterated along the stick (i.e., measured the stick with their straw). To improve precision of measurements, they marked end point of each iteration on the stick, making it possible to follow how length accumulated with iterations (Fig. 4b).

(a) (b)

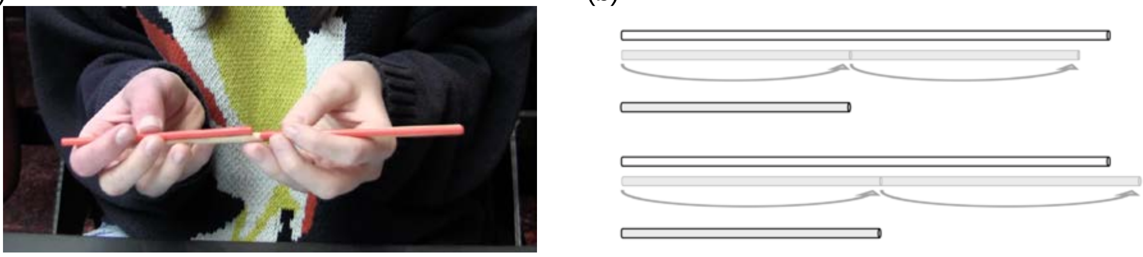

Figure 4. Checking the length of an estimated *small of two* (a) PSTs by creating 2 copies of estimated length, and (b) primary students by iteration of estimated length along the stick.

The latter approach, indeed, was the design intention for activities in which the image of a unit fraction was to be created as a length that fulfils a specific iterative condition with respect to the reference unit. Our analyses of classroom design experiments show that engaging in creating unit fraction lengths via iterations supported primary students in coming to reason proficiently about inverse order relation, and to create and compare any fraction lengths, including those larger than the stick (Cortina & Visnovska 2015). Most importantly, reasoning about these comparisons in terms of iterations of a specific length cultivates multiplicative view of the measurement situation, one that becomes crucial in making reciprocal comparisons (if *A* is *n* times as long as *B*, then *B* must be 1/*n* times as long as *A*), and in making sense of fraction multiplication.

In retrospect, the ways in which PSTs approached the problem of creating the small of two indicates that they keenly translated the measurement problem situation to a familiar one, where a whole was being cut and split into two identical parts. Their 'leaning on familiar' was also evident in PSTs' consistent references to 'half' throughout the session even though the instructor used language of smalls. PSTs were eager to demonstrate what they knew, and encouraging them to develop a different solution proved difficult. Many were not convinced that their approach would not be typical in a year 3 classroom, and the instructor was not prepared to justify convincingly why developing another solution strategy—one based on iterating a small—was a useful way to proceed.

IMPLICATIONS AND CONCLUSIONS

Anticipating tertiary students' approaches to (mathematical) tasks can be difficult, especially when we engage them in activities that are very different from those that they experienced in their prior learning. Understanding how disciplinary ideas can be developed with novice learners did not prepare us sufficiently for supporting PSTs' workshop participation.

Interactions we documented here led us to appreciate specific ways in which PSTs are likely to approach tasks from the *Fractions as Measures* sequence. This awareness made us overall more attuned to anticipating that PSTs are likely to reach for various mathematical tools, even—and perhaps especially—when faced with problems designed for young learners with no requirements for prior mathematical knowledge. We have also learned that restricting acceptable solution strategies by banning use of specific mathematical tools and ideas appears rather arbitrary from PSTs' point of view. They had not yet developed insights into types of solutions that are possible or likely in a primary classroom, but such insights were crucial in order to progress within the sequence of learning activities. In our case, this led to lecturing PSTs on the reality of how students act or do not act in a primary classroom, which PSTs found rather dubious.

The analysis made it clear to us that PSTs need additional support in order to broaden the range of solution strategies that are readily accessible to them, so that it would come to include the (seemingly simpler) solutions that pupils in primary classrooms produce seamlessly. In other words, PSTs need concerted support if they are to overcome limitations to their imagination and creativity that appear to result from their prior mathematical learning. This support needs to involve introduction of primary student solution strategies by means *different* than the instructor's authoritative claims. PSTs would also benefit from opportunities to analyse their 'go to' solution approaches and come to view these as unavailable to young learners who are yet to master the mathematics that is the aim of instruction.

We are re-thinking the fraction sequence workshops in following way: Instead of engaging PSTs in working through the entire learning sequence as students, we are designing tasks to address their *mathematical knowledge for teaching* in the context of the fraction sequence. In place of the workshop activities described in this paper, we would now start by providing PSTs with written or video cases that capture primary student solutions to creating a small of two. We would subsequently ask PSTs to envision how might these same students go about creating a small of three, four, and so on, and how they might reason about relative sizes of smalls.

On a separate, later occasion, we would encourage PSTs to share whether they would themselves approach the task of creating a small of two differently from the case student examples. We are not certain whether, after exploring student approaches, the strategies documented here would remain prevalent among PSTs. If so, we would seek ways to discuss how those strategies reflect prior learning of PSTs, and what might be their limitations.

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