Load on right span.

\[ R_3 = (1 - k) - R_2 \left(1 + \frac{1}{r}\right) \]

\[ = (1 - k) + \frac{r}{2} K_2 \]

\[ k = 1 \]

\[ \int R_3 = \left\{ \frac{1}{2} + \frac{r}{8} \right\} W_3 \]

\[ k = 0 \]

In each case \( R_2 + R_3 + R_4 = \) unity.

---

**Table 5.—Bending Moments and Reactions.**

**Figs. 5a—5d.**

\( l_a = 15. \quad l_b = 12. \quad r = 5/4. \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( M_3 )</th>
<th>( R_2 )</th>
<th>( R_3 )</th>
<th>( R_4 )</th>
<th>( M_3 )</th>
<th>( R_2 )</th>
<th>( R_3 )</th>
<th>( R_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>-0.00</td>
<td>+0.00</td>
<td>+1.00</td>
<td>+0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>+0.00</td>
<td>+1.00</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.06</td>
<td>+0.12</td>
<td>+0.62</td>
<td>+0.98</td>
<td>-0.06</td>
<td>-0.12</td>
<td>+0.12</td>
<td>+0.98</td>
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<td>0.8</td>
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<td>-0.13</td>
<td>-0.21</td>
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<td>+1.67</td>
</tr>
<tr>
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<td>+0.30</td>
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<td>-0.20</td>
<td>-0.30</td>
<td>+0.30</td>
<td>+2.57</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.29</td>
<td>+0.40</td>
<td>+1.45</td>
<td>+3.57</td>
<td>-0.29</td>
<td>-0.40</td>
<td>+0.40</td>
<td>+3.57</td>
</tr>
<tr>
<td>0.5</td>
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<td>+1.76</td>
<td>+4.65</td>
<td>-0.38</td>
<td>-0.50</td>
<td>+0.50</td>
<td>+4.65</td>
</tr>
<tr>
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<td>+2.08</td>
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<td>-0.47</td>
<td>-0.60</td>
<td>+0.60</td>
<td>+5.83</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.56</td>
<td>+0.70</td>
<td>+2.40</td>
<td>+7.12</td>
<td>-0.56</td>
<td>-0.70</td>
<td>+0.70</td>
<td>+7.12</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.64</td>
<td>+0.80</td>
<td>+2.72</td>
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<td>-0.64</td>
<td>-0.80</td>
<td>+0.80</td>
<td>+8.47</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.73</td>
<td>+0.90</td>
<td>+3.05</td>
<td>+9.83</td>
<td>-0.73</td>
<td>-0.90</td>
<td>+0.90</td>
<td>+9.83</td>
</tr>
</tbody>
</table>

Some applications will be given of the above.

**B. Mt. at 3 due to left span fully loaded with uniform load \( w \) per unit of length, Fig. 5e.**

\[
\begin{align*}
\frac{1}{2} \int_0^{l_2} (k - k^2) dk \times w l_2 &= \frac{l_2}{2(1 + r)} \int_0^{1} (k - k^2) dk \times w l_2 \\
&= \frac{wl_2^2}{8(1 + r)} = -\frac{W_2 l_2}{8(1 + r)} \quad \text{where} \ W_2 \text{ is total load on } l_2
\end{align*}
\]

**B. Mt. at 3, right span fully loaded. (Fig. 5f.)**

\[
\begin{align*}
\frac{1}{2} \int_0^{l_3} \left\{ (1 - k) - (1 - k)^3 \right\} d(1 - k) l_3 w &= \frac{l_3 r}{2(1 + r)} \int_0^{1} \left\{ (1 - k) - (1 - k)^3 \right\} d(1 - k) l_3 w
\end{align*}
\]
\[ \frac{-wl^3r}{8(1+r)} = \frac{-W_3l^3r}{8(1+r)} \text{ where } W_3 \text{ is total load on } l_3 \]

The latter result may be obtained from the former by putting \( r = \frac{1}{r}, W_3 \text{ for } W_2 \) and \( l_3 \text{ for } l_2 \).

These examples will serve to illustrate the meaning of the integral sign and the method of deducing the results given above.

\[ R_1 + R_2 + R_3 = W_2 \text{ or } W_3 \text{ according to whether load is as in Fig. 5c or Fig. 5f.} \]

The diagram for \( M_3 \) Fig. 5d, shews graphically how Clapeyron's equation is connected with the general equation for a concentrated load, and the last results all check with results worked directly from Clapeyron's equation.

By means of these diagrams the points of contravlexure (where B. Mt. is zero), and also shears may be readily deduced.

---

**Case 6.—THREE SPANS ON FOUR SUPPORTS, THE SPANS BEING EQUAL IN LENGTH. (Fig. 6.)**

![Diagram](image)

**Bending moment at 2 (Fig. 6a.)**

Load on span, \( l_1 \):

\[ M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = -l_1^2 (K_1). \]

Here \( M_1 = 0 \) and \( l_1 = l_2 = l \) say.

\[ 4M_2 + M_3 = -lK_1 \text{ considering 1, 2, 3 supports} \]

\[ M_2 + 4M_3 = 0 \text{ ,, , , 2, 3, 4 ,,} \]

\[ \therefore M_2 = -\frac{4l}{15} K_1 \text{ and } M_3 = \frac{l}{15} K_1. \]

Load on span \( l_2 \):

\[ 4M_2 + M_3 = -lK_2 \text{ considering 1, 2, 3 supports} \]

\[ M_2 + 4M_3 = -lK_1 \text{ ,, , , 2, 3, 4 ,,} \]

\[ \therefore M_2 = -\frac{l}{15} \left\{4K_2 - K_1\right\} \quad M_3 = -\frac{l}{15} \left\{4K_1 - K_2\right\} \]

Load on span \( l_3 \):

By symmetry \( M_2 \) for load on \( l_3 \) is same as \( M_3 \) for load on \( l_1 \) with \( (1 - k) \) put for \( k \).

\[ M_2 = \frac{l}{15} K_2 \]

The influence line of B. Mt. at 3 is deduced by symmetry from \( M_2 \).

Reaction \( R_1 \).
By similar reasoning to Case 4.

Load on end span \( l_1 \)
\[
R_1 = \frac{M_2}{l} + (1 - k) = (1 - k) - \frac{4}{15}K_1
\]
\[
k = 1
\]
\[
\int R_1 = \frac{13}{30} \text{ W}
\]
\[
k = 0
\]

Load on mid. span \( l_2 \)
\[
R_1 = \frac{M_2}{l} = -\frac{1}{15}(4K_2 - K_1)
\]
\[
k = 1
\]
\[
\int R_1 = -\frac{1}{20} \text{ W}
\]
\[
k = 0
\]

Load on end span \( l_3 \)
\[
R_1 = \frac{M_2}{l} = \frac{1}{15}K_1
\]
\[
k = 1
\]
\[
\int R_1 = \frac{+1}{60} \text{ W}
\]
\[
k = 0
\]

Reaction \( R_2 \)

Load on \( l_1 \)
\[
R_2 = \frac{M_3}{l} + (2 - k) - 2R_1 = \frac{9}{15}K_1 + k.
\]
\[
k = 1
\]
\[
\int R_2 = \frac{13}{20} \text{ W}
\]
\[
k = 0
\]

Load on \( l_2 \)
\[
R_2 = \frac{M_3}{l} + (1 - k) - \frac{5}{2}R_1 = (1 - k) + \frac{1}{5}(3K_2 - 2K_1)
\]
\[
k = 1
\]
\[
\int R_2 = \frac{+11}{20} \text{ W}
\]
\[
k = 0
\]

Load on \( l_3 \)
\[
R_2 = \left\{ (1 - k) - 3R_1 - R_3 \right\} \frac{1}{2} = -\frac{6}{15}K_2
\]
\[
k = 1
\]
\[
\int R_2 = \frac{-1}{10} \text{ W}
\]
\[
k = 0
\]

\( R_3 \) is deduced from \( R_2 \) by symmetry, and \( R_4 \) is deduced from \( R_1 \) in same way.
Table 6.—Shewing Bending Moments and Reactions for Three Spans on Four Supports.

<table>
<thead>
<tr>
<th>Load on $l_1$</th>
<th>Load on $l_2$</th>
<th>Load on $l_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$M_2$</td>
<td>$R_1$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0.05</td>
<td>-0.0133</td>
<td>0.0368</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.0264</td>
<td>-0.0736</td>
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<tr>
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<td>-0.0391</td>
<td>0.8109</td>
</tr>
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<td>-0.0512</td>
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</tr>
<tr>
<td>2.5</td>
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<tr>
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<td>-0.6272</td>
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</tr>
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<td>0.5104</td>
</tr>
<tr>
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</tr>
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<td>0.0000</td>
</tr>
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<td>5.5</td>
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<td>-0.3477</td>
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</tr>
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</tr>
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<td>0.0253</td>
</tr>
<tr>
<td>1.00</td>
<td>-0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

It will be seen from the above that the design of a continuous girder, so far as stresses are concerned, is as direct as that of a plain girder. The equations and theory used depend on the supports moving uniformly (it is beyond the limit of this paper to investigate the effects of supports not moving uniformly), and have all the liability to error due to imperfect elasticity and such other factors as operate against a definite knowledge of the stresses in any structure which is not of the "simple" kind; but for the finding of stresses according to the ordinary continuous beam theory, the writer thinks that the method is clearer and neater than other methods.

The curves for reactions need only be plotted for the spans under consideration, and the reactions for any loading are readily deduced from the tables (or the value of the ordinates might be written on them if there were room). From these the bending moments and shears are readily deduced.

For instance, in the working out of a continuous girder, Warren's Engineering Construction, pp. 168-175, in getting the maxima bending moments, five cases are taken and the moments worked out for each; in the method used there the calculations could be simplified by the use of influence lines to give the reactions, and the bending moments and shears worked directly.
In short, the Influence Lines work out the required function once for all for the structure; while without them much of the same ground is covered several times.

Take Case 1, in example mentioned above (Fig. 6d).

\[ l_1 \text{ loaded with live and dead loads } = 1.3 \text{ tons per ft.} \]
\[ l_2, l_3 \text{ " dead load only } = 0.6 \text{ " " } \]
\[ l_1 = l_2 = l_3 = 159 \text{ ft.} \]

Thus \[ M_1 = - \frac{1}{15} W_a - \frac{1}{20} W_\beta + \frac{1}{60} W_\gamma. \]

Where \[ W_a = \text{ load on span } 12 = 159 \text{ ft. } \times 1.3 \text{ tons per ft.} \]
\[ W_\beta = " " 23 = 159 \text{ ft. } \times 0.6 " " \]
\[ W_\gamma = " " 34 = 159 \text{ ft. } \times 0.6 " " \]
\[ M_1 = B. \text{ Mt. at first pier from left.} \]

Thus \[ M_1 = 159 \left\{ - \frac{1}{15} (1.3) - \frac{1}{20} (0.6) + \frac{1}{60} (0.6) \right\} \]
\[ = 2696.6 \text{ as in text.} \]

This saves the labour and liability to error in deducing and solving the simultaneous equations.

For same case.

\[ R_1 = \frac{13}{30} W_a - \frac{1}{20} W_\beta + \frac{1}{60} W_\gamma. \]
\[ = 159 \text{ ft. } \left\{ \frac{13}{30} \times 1.3 \text{ tons } - \frac{1}{20} \times 0.6 \text{ tons } + \frac{1}{60} \times 0.6 \text{ tons } \right\} \]
\[ = + 86.39 \text{ tons.} \]

\[ R_2 = \frac{13}{20} W_a + \frac{11}{20} W_\beta - \frac{1}{10} W_\gamma. \]
\[ = + 177.285 \text{ tons.} \]

\[ R_3 = \frac{13}{20} W_\gamma + \frac{11}{20} W_\beta - \frac{1}{10} W_a \]
\[ = + 93.81 \text{ tons.} \]

\[ R_4 = \frac{13}{30} W_\gamma - \frac{1}{20} W_\beta + \frac{1}{60} W_a \]
\[ = 40.015 \text{ tons.} \]

We have now the loads and reactions as shown in Fig. 6f. From this can be deduced the B. Mt. or shear at any point and the full effect of the loading known in detail as with a simple beam.

The other cases could be similarly treated, and, if required, influence lines of B. Mt. or Shear at any point could be constructed.

It is stated on p. 175, Warren’s Engineering Construction. “In consequence of the change of position of the points of contraflexure during the passage of a rolling load, the shearing stresses cannot be accurately determined without considerable labour,” and the text shews approximate methods of dealing with moving loads, distributed and concentrated.
With diagrams such as Figs. 6a, 6b, 6c drawn to scale, the shears for any position of a rolling load offer no difficulty.

An example will be given. (Figs. 6ε and 6f.)

Take a rolling load, \( W_1, W_2 \) on span 12.
Shear at any point A on span 12.
\[
\begin{align*}
\text{Shear} & = \left\{ W_1 \times PM \right\} + \left\{ W_2 \times P_1 M_1 \right\} - (W_1 + W_2) \quad \text{Fig. 6ε.}
\end{align*}
\]
Again, shear at any point B on span 23. (Figs. 6ε and 6f.)
\[
\begin{align*}
\text{Shear} & = W_1 \times \left\{ PM + P_1 M_1 \right\} + W_2 \left( P_1 M_1 + P_1 M_1 \right) - (W_1 + W_2)
\end{align*}
\]
Take the loads to have rolled on span 23.

Shear at A \(-\)
\[
\begin{align*}
\text{Shear} & = -\left\{ W_1 \left( PM \right) + W_2 \left( P_1 M_1 \right) \right\} \quad \text{Fig. 6ε. Span 23.}
\end{align*}
\]
Shear at B \(-\)
\[
\begin{align*}
\text{Shear} & = -\left\{ W_1 \left( PM \right) + W_2 \left( P_1 M_1 \right) \right\} + \left\{ W_1 \left( P_1 M_1 \right) + W_2 \left( P_1 M_1 \right) \right\} - (W_1 + W_2)^* \quad \text{Figs. 6ε and 6f. Span 23.}
\end{align*}
\]
This example will suffice to shew the method of working. Results could be similarly deduced for concentrated or distributed loads in any position.

---

**Case 7.—THREE SPANS, THE TWO END SPANS BEING EQUAL IN LENGTH.** (Fig. 7.)

Let \( l_1 = l_3 = l \) say.
and \( l_2 = nl \), where \( n \) may be \( \leq \) unity.

Let \( m \) represent the quantity \( 4 \left(1 + n\right)^2 - n^2 \) which will be often used.

As before \( K_1 = k - k^3 \) and \( K_2 = (1 - k) - (1 - k)^3 \)

**Bending Moments.**

From general equation \( M_2 l_2 + 2M_3 \left(l_2 + l_3\right) + M_4 l_3 = -l_2^2 \left(k - k^3\right) \)

\* The last term non-existent when B is on left of M.
Load on \( l_1 \). Since \( M_1 = M_4 = \text{zero} \).
\[ 2M_2 (l_1 + l_2) + M_3 l_2 = -l_1^2 K_1 \]
also \( M_2 l_2 + 2M_3 (l_2 + l_3) = 0 \)
\[ \therefore M_2 = \frac{2 (1 + n)}{m} K_1 l \text{ and } M_3 = \frac{n K_1 l}{m} \]

Load on \( l_2 \).
\[ 2M_2 (l_1 + l_2) + M_3 l_2 = -l_2^2 K_2 \]
\[ M_2 l_2 + 2M_3 (l_2 + l_3) = -l_2 K_1 \]
\[ \therefore M_2 = \frac{-ln^2}{m} \left[ 2 (1 + n) K_2 - n K_1 \right] \]
and \[ M_3 = \frac{-ln^2}{m} \left[ 2 (1 + n) K_1 - n K_2 \right] \]

Load on \( l_3 \). By symmetry.
\[ M_2 = + \frac{n}{m} K_2 l \text{ and } M_3 = -\frac{2 (1 + n)}{m} K_2 l \]

Thus for Influence Lines of \( M_2 \)

Load on \( l_1 \). \( M_2 = \frac{-2 (1 + n)}{m} K_1 l \int M_2 = \frac{- (1 + n)}{2m} W l. \]
\[ \therefore k = 1 \]
\[ \int M_2 = \frac{-n^2}{m} - \frac{2 + n}{4} W l. \]
\[ \therefore k = 0 \]

Load on \( l_2 \). \( M_2 = \frac{n^2}{m} \left\{ 2 (1 + n) K_2 - n K_1 \right\} l \)
\[ \int M_2 = \frac{n}{4m} W l. \]
\[ \therefore k = 0 \]

Load on \( l_3 \). \( M_2 = \frac{n}{m} K_2 l \)
\[ \int M_2 = \frac{n}{4m} W l. \]
\[ \therefore k = 0 \]

The curve is shown in Fig. 7a, see also Table 7.

Reaction \( R_1 \).
Load on \( l_1 \). \( R_1 = \frac{M_2}{l} + (1 - k) = (1 - k) - \frac{2 (1 + n)}{m} K_1. \)
\[ \int R_1 = \left\{ \frac{1}{2} - \frac{1 + n}{2m} \right\} W. \]
\[ \therefore k = 1 \]
\[ \int R_1 = \frac{-n^2}{m} \cdot \frac{2 + n}{4} W. \]
\[ \therefore k = 0 \]

Load on \( l_2 \). \( R_1 = \frac{M_2}{l} = -\frac{n^2}{m} \left\{ 2 (1 + n) K_2 - n K_1 \right\} \)
\[ \int R_1 = -\frac{n^2}{m} \cdot \frac{2 + n}{4} W. \]
\[ \therefore k = 0 \]
Load on $l_3$. $R_1 = \frac{M_3}{l} = \frac{n}{m} K_2$. 

\[ k = 1 \]

\[ \int R_1 = \frac{n}{4m} W. \]

\[ k = 0 \]

The curve is shown in Fig. 7b, see also Table 7.

Reaction $R_2$.

Load on $l_1$. $R_2 = \left\{ \frac{M_3}{l} + (1 + n - k) - R_1 (1 + n) \right\} \frac{1}{n}$.

\[ k = 1 \]

\[ \int R_2 = \left\{ \frac{1}{2} + \frac{1}{4} \left\{ \frac{2 (1 + n)^2 + n}{m n} \right\} \right\} W. \]

\[ k = 0 \]

Load on $l_2$. $R_2 = \left\{ \frac{M_3}{l} + (1 - k) n - R_1 (1 + n) \right\} \frac{1}{n}$.

Where $\frac{M_2}{l} = R_1$.

\[ = \frac{M_3 - M_2}{ln} - \frac{M_2}{l} + (1 - k) n. \]

\[ = (1 - k) + \frac{n}{m} \left\{ \frac{2 (1 + n)^2 + n}{m n} \right\} K_2 - (1 + n) (2 + n) K_1 \]

\[ k = 1 \]

\[ \int R_2 = \left\{ \frac{1}{2} + \frac{n}{4m} \left( \frac{2n + n^2}{4m} \right) \right\} W. \]

\[ k = 0 \]

Load on $l_3$. $R_3 = \frac{M_3}{ln} - \frac{R_1 (1 + n)}{n}$

\[ = - \frac{(1 + n) (2 + n)}{m n} K_2. \]

\[ k = 1 \]

\[ \int R_2 = - \frac{(1 + n) (2 + n)}{4 m n} W. \]

\[ k = 0 \]

On account of symmetry, $R_3$ and $R_4$ will be same as $R_2$ and $R_1$ respectively, taking $k$ from right end instead of from left.
Table 7.—Shewing Bending Moments and Reactions for Case 7.

Figures are worked out for 11 = 5/4.

<table>
<thead>
<tr>
<th>Load on l₁</th>
<th>Load on l₂</th>
<th>Load on l₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>M₂</td>
<td>R₁</td>
</tr>
<tr>
<td>1/5</td>
<td>0.120</td>
<td>9.380</td>
</tr>
<tr>
<td>1/10</td>
<td>0.258</td>
<td>8.762</td>
</tr>
<tr>
<td>1/15</td>
<td>0.353</td>
<td>8.147</td>
</tr>
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<td>0.462</td>
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</tr>
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---

Case 8.—THREE UNEQUAL SPANS. (Fig 8).

Let 1 = n₁ l₁.

l₂ = n₂ l₂.

l₃ = n₃ l₃.

Let 4 (n₁ + n₂) (n₁ + n₂) = n²₂ = m₂.

Bending Moments.

Load on l₁. M₁ l₁ + 2M₂ (l₁ + l₂) + M₃ l₂ = − l₁² (K₁).

Here M₁ = 0 l₁ = n₁ l₁ l₂ = n₂ l₂.

∴ 2M₂ (n₁ + n₂) + M₃ n₂ = − n₁² l K₁. .............. (i.)

also M₂ l₂ + 2M₃ (l₂ + l₃) + M₄ l₃ = 0.

∴ M₃ n₃ + 2M₃ (n₂ + n₃) = 0. ...................... (ii.)

∴ M₂ = − 2n₁² (n₂ + n₃) m₃ K₁ l and M₃ = n₃² n₂ m₂ K₁ l.
Load on \( l_2 \): \[ 2M_2 (n_1 + n_2) + M_3 n_2 + n_2^2 l K_2 = 0. \]
\[ M_3 n_2 + 2 M_3 (n_2 + n_3) + n_2^2 l K_1 = 0. \]
\[ \therefore M_2 = -\frac{n_2^2 l}{m_2} \left[ 2 (n_2 + n_3) K_2 - n_2 K_1 \right] \]
and \[ M_3 = -\frac{n_2^2 l}{m_2} \left[ 2 (n_2 + n_1) K_1 - n_2 K_2 \right] \]

Load on \( l_3 \): \( M_2 \) similar to \( M_3 \) in 1st Case with \( K_2 \) for \( K_1 \).
\[ M_3 \] , \( M_2 \) , \( K_1 \) for \( K_2 \).
In both cases \( n_3 \) and \( n_1 \) are interchanged.
\[ \therefore M_2 = +\frac{n_3^2}{m_2} n_2 K_2 l \text{ and } M_3 = -\frac{2 n_3^2 (n_2 + n_1)}{m_2} K_2 l. \]

Reaction \( R_1 \).

Load on \( l_1 \): \[ R_1 = \frac{M_2}{l_1} + (1 - k) = (1 - k) - \frac{2 n_1 (n_2 + n_3)}{m_2} K_1. \]

Load on \( l_2 \): \[ R_1 = \frac{M_2}{l_1} = -\frac{n_2^2}{n_1 m_2} \left[ 2 (n_2 + n_3) K_2 - n_2 K_1 \right] \]

Load on \( l_3 \): \[ R_1 = \frac{M_2}{l_1} = \frac{n_3^2}{n_1 m_2} K_2. \]

Reaction \( R_2 \).

Load on \( l_1 \): \[ R_2 = \frac{M_3}{l_2} + \frac{l_1 + l_2 - k l_1}{l_2} - R_1 \frac{l_1 + l_2}{l_2} \]
\[ = k + K_1 \left\{ \frac{n_1^2}{n_2 m_2} + 2 n_1 (n_1 + n_2) (n_2 + n_3) \right\} \]

Load on \( l_2 \): \[ R_2 = \frac{M_3}{l_2} + (1 - k) - R_1 \frac{l_1 + l_2}{l_2} \]
\[ = (1 - k) + \frac{n_2}{m_2} \left\{ n_1 (n_1 + n_2) (n_2 + n_3) \right\} K_2.
\[ \quad - (n_1 + n_2) \left\{ \frac{n_2}{n_1} + 2 \right\} K_1 \]

Load on \( l_3 \): \[ R_3 = \frac{M_3}{l_2} - R_1 \frac{l_1 + l_2}{l_2} \]
\[ = -\frac{n_3^2}{m_2 n_2} (n_1 + n_2) (2 n_1 + n_2) K_2. \]

Reaction \( R_3 \).

Similar to \( R_2 \) with 1 put for 3 throughout.
\( k \) put for \( 1 - k \).
\( K_1 \) put for \( K_2 \).

Load on \( l_1 \): \[ R_3 = -\frac{n_3^2 (n_3 + n_2) (2 n_3 + n_2)}{m_2 n_2} K_1. \]

Load on \( l_2 \): \[ R_3 = k + \frac{n_3}{m_2} \left\{ n_3 (n_3 + n_2) (n_1 + n_2) \right\} K_1.
\[ \quad - (n_3 + n_2) \left( \frac{n_3}{n_1} + 2 \right) K_2 \]
Load on $l_3$.

$$ R_3 = (1 - k) + K_2 \left\{ \frac{n_3^2 n_2 + 2n_3 (n_1 + n_2) (n_2 + n_3)}{m_2 n_2} \right\} $$

Reaction $R_4$.

Similar to $R_1$ with $n_3$ for $n_1$ and $K_2$ for $K_1$ and vice-versa.

Load on $l_1$.  

$$ R_4 = \frac{+ m_1^2 n_2}{m_3 m_2} K_1. $$

Load on $l_2$.  

$$ R_4 = \frac{- n_2^2}{m_3 m_2} \left\{ 2 (n_2 + n_1) K_1 - n_2 K_2 \right\} $$

Load on $l_3$.  

$$ R_4 = k - \frac{2 (n_3) (n_2 + n_1)}{m_2} K_2. $$

The curves are similar in shape to Case 7, in fact Case 7 may be deduced from Case 8, by putting $n_1 = n_3 = 1$ and $n_2 = n$.

As a practical example of Case 8, take a beam supporting a building with supports 12 feet, 15 feet and 18 feet apart.  Fig. 8a.

Here $n_1 = 4$, $n_2 = 5$, $n_3 = 6$, and $l = 3$.

Then $m_2 = 4 (n_1 + n_2) (n_2 + n_3) - n_2^2 = 371$.

Reaction $R_1$.

For load on $l_1$.  

$$ R_1 = (1 - k) - \frac{88}{371} K_1. $$

$$ l_2. \quad = - \frac{25}{1484} \left\{ 22 K_2 - 5 K_1 \right\} $$

$$ l_3. \quad = + \frac{45}{371} \left\{ K_2 \right\} $$

Reaction $R_2$.

For load on $l_1$.  

$$ R_2 = k + \frac{872}{1855} K_1. $$

$$ l_2. \quad = (1 - k) + \frac{5}{1484} \left\{ 3248 K_2 - 117 K_1 \right\} $$

$$ l_3. \quad = - \frac{585}{371} K_2. $$

Reaction $R_3$.

For load on $l_1$.  

$$ R_3 = - \frac{2992}{1855} K_1. $$

$$ l_2. \quad = k + \frac{5}{2226} \left\{ 1308 K_1 - 187 K_2 \right\} $$

$$ l_3. \quad = (1 - k) + \frac{1368}{1855} K_2. $$

Reaction $R_4$.

For load on $l_1$.  

$$ R_4 = + \frac{40}{1113} K_1. $$

$$ l_2. \quad = - \frac{25}{2226} \left\{ 18 K_1 - 5 K_2 \right\} $$

$$ l_3. \quad = k - \frac{108}{371} K_2. $$

The fractions above can be put into decimals and the curves traced.
Table 8.—Reactions for Beams, 12', 15', 18' Spans Continuous.

<table>
<thead>
<tr>
<th>k</th>
<th>LOAD ON R1</th>
<th>LOAD ON R2</th>
<th>LOAD ON R3</th>
<th>LOAD ON R4</th>
</tr>
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<tbody>
<tr>
<td>l_1</td>
<td>l_2</td>
<td>l_3</td>
<td>l_1</td>
<td>l_2</td>
</tr>
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</tr>
</tbody>
</table>

Case 9.—FOUR EQUAL SPANS ON FIVE SUPPORTS. (Fig 9).

These might be the stringers of a bridge, or a large beam in a building.

From general equation.

\[
M_2 l_2 + 2M_3 (l_2 + l_3) + M_4 l_3 = -l^2 K_1 \\
\text{since } l_1 = l_2 = l_3 = l_4 = l \text{ say.}
\]

\[
M_2 + 2M_3 + M_4 = -l K \text{ where } K \text{ may be } K_1 \text{ or } K_2
\]

according to which span the load is on.

Bending moments.

Load on \( l_1 \).

\[
\begin{align*}
M_1 + 4M_2 + M_3 &= l K_1 \\
M_2 + 4M_3 + M_4 &= 0 \\
M_3 + 4M_4 + M_5 &= 0
\end{align*}
\]

\[
\therefore M_2 = -\frac{15}{56} l K_1 \quad M_3 = + \frac{1}{14} l K_1 \quad M_4 = -\frac{1}{56} l K_1.
\]

Load on \( l_2 \).

\[
4M_2 + M_3 = -l K_2.
\]

\[
M_2 + 4M_3 + M_4 = -l K_1.
\]

\[
M_3 + 4M_4 = 0.
\]

\[
\therefore M_2 = -\frac{l}{14} \left( \frac{15}{4} K_2 - K_1 \right) \quad M_3 = -\frac{l}{14} \left( 4K_1 - K_2 \right) \quad M_4 = \frac{l}{56} \left( 4K_1 - K_2 \right).
\]
Fig. 9

Fig. 9a. B.M. at 2

Fig. 9b. B.M. at 3.

Fig. 9c. Reaction at 1

Fig. 9d. Reaction at 2.

Fig. 9e. Reaction at 3.

Fig. 10

Fig. 10a.

Fig. 10b. B.M. at 2

Fig. 10c. B.M. at centre

Fig. 10d. Reaction at 2

Fig. 10e. Reaction at 3

Fig. 11

Fig. 11a.

Fig. 11b. B.M. at 2

Fig. 11c. Reaction at 2

Fig. 11d. Reaction at 3

Fig. 11e. B.M. at centre