THE METHOD OF TWO ORIGINS FOR DEDUCING THE DEFLECTION OF A BEAM AND THE EQUATION OF THREE MOMENTS

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The proofs given below were evolved in 1912 and shewn to the students of the time, when the author was giving a portion of the lectures in Civil Engineering at the University of Sydney, during the absence of Professor Warren.

Later it was found that an analogous device to that used for shortening the proofs had been used by Professor Morley.* As the method was a rediscovery only, it was not published, but as it does not seem to be well-known, modern books† still preserving the older form of analysis, some useful purpose may be served by submitting it to this Society.

The ordinary "analytical" method is used generally though the semi-graphical methods shewn in Goodman's "Applied Mechanics," Fidler's "Bridge Construction" may be applied to check results.

Proving the Equation of Three Moments may need some apology as an Engineer is not usually asked to do so, but it will be admitted that its limits of application, and correct manipulation cannot be fully understood unless its basis and the assumptions made are kept in mind. As one instance of this it will be seen that all the subsequent deductions, which are the usual standard results, take account of Bending Moment effects only, though in certain cases shear effects may be appreciable.

Assumptions as to Elasticity, Supports, etc., are indicated in the text.

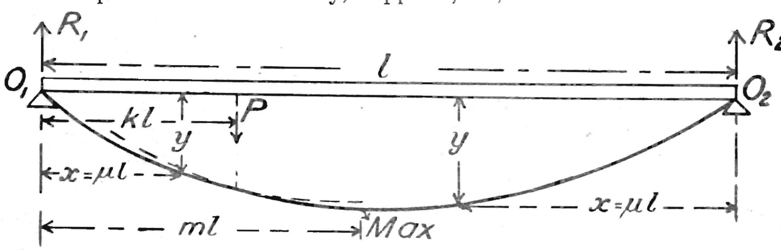

Fig. 1

Deflection of a Simply-Supported Beam due to a Single Load Unsymmetrically Placed.

Referring to Figure 1.

Let P be a load at the point P.

l be length between supports O_1, O_2.

* Morley.—"Theory of Structures." Mr. Mansfield Merriman writes under date Dec. 27th, 1913. "Your method of using two origins for the discussion of deflection and the Theorem of three moments is an excellent one which much abridges the algebraic work." Something similar I have seen in German books, but cannot now give you references."

† Andrews.—"Theory and Design of Structures."

† Merriman.—"Mechanics of Engineering."

kl be distance of load from left support.
ρ be radius of curvature at any point.
M be Bending Moment at any point.
E be Modulus of Elasticity at any point.
I be Moment of Inertia at any point.
R_1 and R_2 be Reactions at ends.
y, x be co-ordinates of points on the deflection curve, with origins as indicated below in the text, later μl is written for x.
C_1, C_2, etc., be Constants.
$K_1 = k - k^3 = k(1-k)(1+k)$.
$K_2 = \left\{ (1-k) - (1-k)^3 \right\} = k(1-k)(2-k)$.
$k_1 l$ = distances of load from other support to that which is chosen as origin.
ml = distance of point of max. deflection from one end.
μl = distance of point from origin, i.e., abscissa of any point.
If E or I vary plot $\frac{M}{EI}$ at each point in solving practical examples (see appendix).

By the principles of elasticity it is proved*

$$\frac{d^2y}{dx^2} = \frac{1}{\rho} = \frac{M}{EI} \text{ and } \frac{dy}{dx} = \int \frac{M}{EI} dx + C_1.$$

$$y = \int \int \frac{M}{EI} dx\, dx + C_1 x + C_2.$$

Now $R_1 = P(l-k)$.
Looking to Left of P with *origin at O_1 axis of x being $O_1 O_2$*.
$M = R_1 x$ up to the load.

$$\therefore EI \frac{dy}{dx} = \int M\, dx = R_1 \frac{x^2}{2} + C_1 \quad \ldots \quad \ldots \quad (1)$$

$$EI\, y = \int \int M\, dx\, dx = R_1 \frac{x^3}{6} + C_1 x + C_2.$$

now when $x = 0$ $y = 0$ $\therefore C_2 = 0$.
$\therefore$ Equation of curve of deflection up to the load

$$\text{is } EI\, y = R_1 \frac{x^3}{6} + C_1 x \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Again $R_2 = P(k)$.
Looking to right of P, with *origin at O_2 axis of x being $O_2 O_1$*.
$M = R_2 x$ up to the load.

$$\therefore EI \frac{dy}{dx} = \int M\, dx = R_2 \frac{x^2}{2} + C_3 \quad \ldots \quad \ldots \quad (3)$$

$$EI\, y = \int \int M\, dx\, dx = R_2 \frac{x^3}{6} + C_3 x + C_4.$$

When $x = 0$ $y = 0$ $\therefore C_4 = 0$.
$\therefore$ Equation of curve of deflection up to the load

$$\text{is } EI\, y = R_2 \frac{x^3}{6} + C_3 x \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

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* Warren "Engineering Construction." p. 193, etc.

Equation (4) could be written from Equation (2) at once by the consideration that if we imagined we were looking through the back of the paper the point O_2 becomes O_1, and *vice versa*. However it was thought better to write it in full as probably what follows will be easier understood.

At the point P, *i.e.*, under the load the $\frac{dy}{dx}$ and y respectively must be the same in amount whether derived from Equations (1) and (2) or Equations (3) and (4), but the *sign* of $\frac{dy}{dx}$ is of opposite sign, since as x increases in (1) it decreases in (3).

This use of *two origins* is where the present proof differs from previous methods : it will be seen that it eliminates one constant (C_4 would not be zero if equation (4) were to origin O_1), thus saving considerable labour in deducing the constants.

At P in (1) $x = kl$. In (3) $x = (1 - k)\, l$.

$R_1 = P\,(1 - k)$. $R_2 = P\,(k)$.

$$\therefore\ P\,(1 - k)\,\frac{k^2 l^2}{2} + C_1 = -\,P\,(k)\,\frac{(1-k)^2\, l^2}{2} - C_3 \text{ from (1) and (3)} \quad \ldots \quad (5)$$

$$\text{And } P\,(1 - k)\,\frac{k^3\, l^3}{6} + C_1\, kl = P\,k\,\frac{(1 - k)^3\, l^3}{6} + C_3\,(1 - k)\, l \text{ from (2) and (4)} \quad \ldots \quad (6)$$

Equations (5) and (6) are simultaneous equations to deduce C_1 and C_3.

Multiply (5) by $(1 - k)\, l$ and add to (6). After collecting terms we get $C_1 = -\,\frac{P l^2}{6}\, k\,(1 - k)\,(2 - k) = -\,\frac{P l^2}{6}\, K_2$ (7)

by putting $k = (1 - k)$ in (5) by reasoning as above, or by solving the simultaneous equations.

$$C_3 = -\,\frac{P l^2}{6}\, k\,(1 - k)\,(1 + k) = -\,\frac{P l^2}{6}\, K \quad \ldots \quad (8)$$

Equation (2) *i.e.*, equation of curve of deflection from O_1 to P may be written with *origin at* O^1.

$$\therefore\ \frac{6\,EI}{P}\, y = (1 - k)\, x^3 - l^2\, K_2\, x\ \dagger \quad \ldots \quad \ldots \quad (9)$$

Equation (4) *i.e.*, equation of curve of deflection from O_2 to P may be written with *origin at* O_2.

$$\frac{6\,EI}{P}\, y = k\, x^3 - l\ K_1\, x\ \dagger \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

† For table of K_1 and K_2 see Merriman & Jacoby "Higher Structures." Part IV., p. 35, quoted in the author's paper on "Influence Lines." Proc. S.U.E.S., 1903; also in Warren's "Engineering Construction," p. 233.

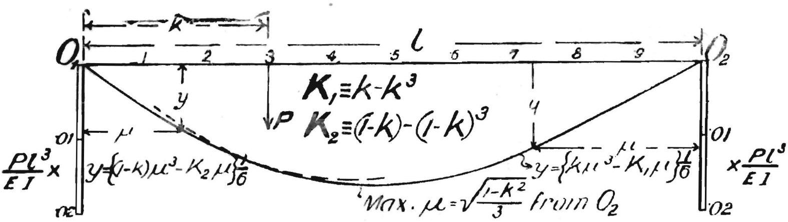


Fig. 1A

It will be seen from (9) and (10) and Figure 1A that the curve of deflection changes its equation under the load though curves are tangential; this may be seen readily by putting $(1 - x)$ for x in (10), we would get the equation to origin at O_1 such equation not being the same as (9). The method of analysis also shows such to be the case. We may write (9) and (10) in the form

$$\frac{6\,EI}{P}\,y = k_1 x^3 - l^2 (k_1 - k_1^3)\,x.$$

Where $k_1 l$ is distance from *other* end to that which is chosen as origin.

Or put in neater form using μ as the coefficient lengths, μl being written for x.

$$y = \left\{\frac{Pl^3}{EI}\right\} \left\{\frac{1}{6}\right\} \left\{k_1 \mu^3 - (k_1 - k_1^3)\,\mu\right\} \quad \ldots \quad (11)$$

for instance at point ·3 from left end.

Equations of curves of deflection are since $k_1 = \cdot 7$.

$$\left.\begin{array}{l}\text{From } O_1 \text{ to Load}\\ \text{origin at } O_1\end{array}\right\} y = \frac{Pl^3}{EI}\,\frac{1}{6}\left\{\cdot 7\,\mu^3 - (\cdot 3570)\,\mu\right\}.$$

$$\left.\begin{array}{l}\text{From } O_2 \text{ to Load}\\ \text{origin at } O_2\end{array}\right\} y = \frac{Pl^3}{EI}\,\frac{1}{6}\left\{\cdot 3\,\mu^3 - (\cdot 2730)\,\mu\right\}$$

Figure 1A shews graphically the results obtained and Figure 1B several typical curves.

Points of Maximum Deflection.

We may use equations (1) and (3), *i.e.*, the equations of slopes and maximum deflection is where the slope is zero; or may take (3) and (5), the equations of deflections, and where $\frac{dy}{dx}$ is zero the deflection is a maximum, either course leads to the same results.

Taking the equations of slopes

$$\text{since } R_1 = (1 - k)\,P \text{ and } C_1 = -\frac{K_2}{6}\,P \text{ for unit } l \qquad (12)$$

$$R_2 = k\,P \text{ and } C_3 = -\frac{K_1}{6}\,P \text{ for unit } l \qquad (13)$$

CURVES OF DEFLECTION

BEAM WITH SINGLE LOAD

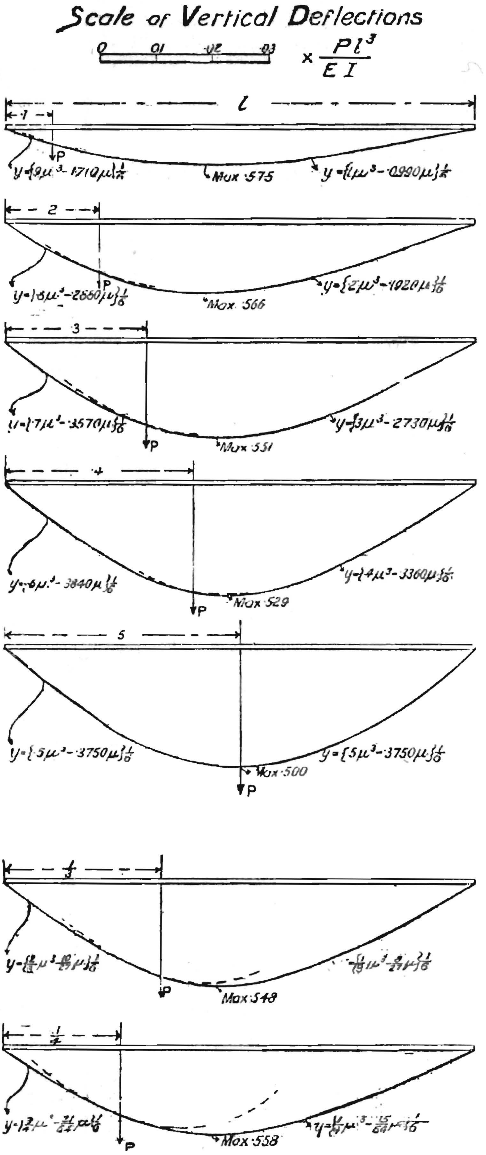


Fig. 1B

we get if ϕ = slope

$$\phi = \frac{Pl^2}{EI}\left\{(1-k)\frac{\mu^2}{2} - \frac{(K_2)}{6}\right\} \text{ from origin } O_1 \text{ to } P$$

and origin at O_1 ... ... (14)

$$\phi = \frac{Pl^2}{EI}\left\{k\frac{\mu^2}{2} - \frac{K_1}{6}\right\} \text{ from origin } O_2 \text{ to } P$$

and origin at O_2 ... ... (15)

If $m_1 l$ = distance of point of max. deflection from O_1
$m_2 l$ = distance of point of max. deflection from O_2

When $\phi = o$ in (14)

$$(1-k)\frac{m_1^2}{2} = \frac{K_2}{6} \text{ or } m_1^2 = \frac{1-(1-k)^2}{3} \quad \ldots \quad (16)$$

When $\phi = o$ in (15)

$$k\frac{m_2^2}{2} = \frac{K_1}{6} \text{ or } m_2^2 = \frac{1-k^2}{3} \quad \ldots \quad \ldots \quad (17)$$

It is readily seen that when $k = \frac{1}{2}$; $m = \frac{1}{2}$.
Writing $k = \frac{1}{2} + dk$ we get in (16).

$$k^2 - m_1^2 = \frac{2\,dk}{3}(1 + 2\,dk).$$

Which is $\left\{\begin{array}{l}\text{negative}\\ \text{positive}\end{array}\right.$ when dk is $\left\{\begin{array}{l}\text{negative}\\ \text{positive}\end{array}\right.$

i.e., $m < k$ when $k > \frac{1}{2}$ and max. point is within the limit of the equation.

When $m > k$, *i.e.*, when $k < \frac{1}{2}$ we are outside the limit of the curve which only applies up to $x = kl$ there being two curves of deflection which are tangential under the load.

Also in (17) $m_2 > k$ to be within limits of the equation,

$$\text{or in (17) } (1-m_1)^2 = \frac{1-k^2}{3} \text{ when } k < \tfrac{1}{2} \quad \ldots \quad (18)$$

$$\text{and in (16) } m_1^2 = \frac{1-(1-k)^2}{3} \text{ when } k > \tfrac{1}{2} \quad \ldots \quad (19)$$

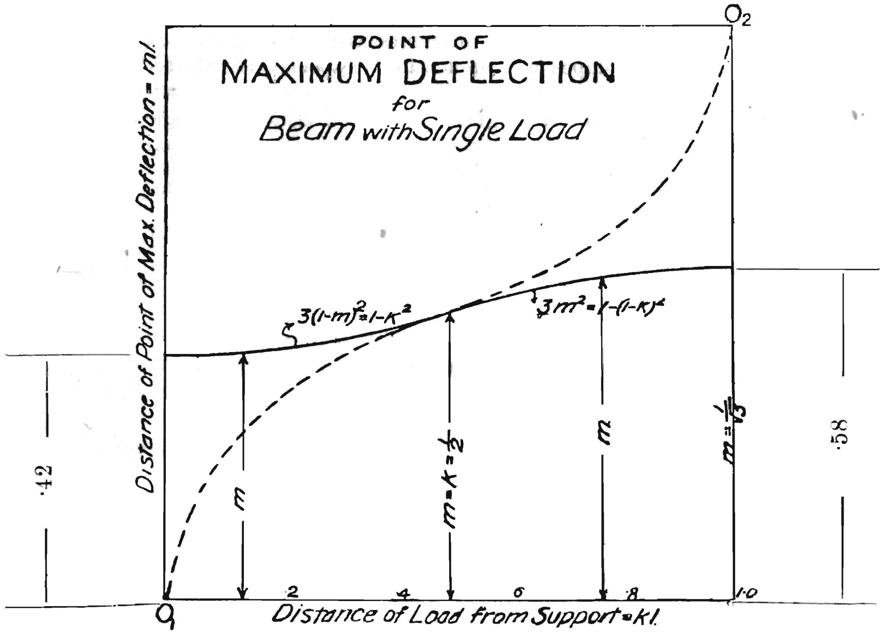
We can drop the suffix in m_1

Thus the equations giving m are

$$(1-m)^2 = \frac{1-k^2}{3} \text{ from } k = O \text{ up to } k = \tfrac{1}{2} \quad \ldots \quad (20)$$

$$m^2 = \frac{1-(1-k)^2}{3} \text{ from } k = \tfrac{1}{2} \text{ to } k = 1 \quad \ldots \quad \ldots \quad (21)$$

These are the equations of ellipses. They show the curious effect that by putting the load ever so small a distance from the origin the point of max. deflection is near the centre, the limit of range being $m = \cdot 42$ to $m = \cdot 58$. The point of max. deflection is always between the point of application of the load and the centre of the beam. So far as the author is aware, the change of the equation of deflection at the load has not been emphasised sufficiently, and using this method there is a liability of error in calculating the point of max. deflection unless the above principles as illustrated by the curve, Fig. 2, are borne in mind.



— *Fig. 2.* —

For instance taking $k = \cdot 3$

If we use equation (9) for curve of deflection we get

$$m = \sqrt{\frac{k\,(2-k)}{3}}$$

as usually written this is the formula which the position of the max. point is found either in this or other notation

$$= \cdot 412.$$

whereas the true value

$$\text{of } m_2 = \sqrt{\frac{\cdot 7\,(1 \cdot 3)}{3}} = \cdot 55$$

$$\text{or } m = 1 - \cdot 55 = \cdot 45.$$

The curve of Fig. 2 shews graphically the results obtained.

Many important facts may be deduced from the study of the equations and diagrams deduced above.

For example :—

From (14) when $x = o$ $\qquad \phi = \frac{K_2}{6}$ giving the slope at O_1

From (15) when $x = o$ $\qquad \phi = \frac{K_1}{6}$ giving the slope at O_2

Thus it may readily be seen that the amount of the slope at the support is equal to the reaction at the same support when the B. Mt. diagram is treated as a load diagram. From this fact is deduced the elegant method for simple beams—load the beam with the B. Mt. diagram and the resultant Shears and Bending Moments (treating the B. Mt. diagram as a load), are the slopes and deflections respectively referred to the line through the supports as zero (this latter is usually horizontal).

Again all the summations give slopes and deflections referred to the tangent at the end from which summation starts; similarly if $\int Mxdx$ is used instead of $\int\int Mdxdx$ the quantity deduced is the deflection referred to the tangent at the other extremity from that at which moments are taken.

It is beyond the limits of this paper to examine the question of deflections in general, but, as an appendix, Fig. 5 is given to show a general case that practically includes perhaps most of the difficulties that occur in connection with beam deflections. The diagrams shown may be used to illustrate most of the principles and results of computing shears, bending moments, slopes and deflections.

Curves for slopes are not given as these are not used much in practice, but they could readily be drawn from equations (14) and (15) if required.

Figure 1B shows the curves of deflection for the typical points; if the load is on the right of the centre the corresponding curves are readily deduced, in fact looking through the back of the paper this may be seen at once.

The Equation of Three Moments for a Continuous Girder.

In a continuous girder the "Equation of Three Moments" connects the bending moments at three contiguous supports with the loads and spans, by the relation to be deduced.

Applications of the use of the equation are illustrated in text-books*, and in the paper by the author on "Influence Lines."†

The proof follows the method used in Merriman & Jacoby's "Higher Structures," Vol. IV., in the analytic method of reasoning; but the introduction of *two* origins shortens the work considerably and practically remodels the proof.

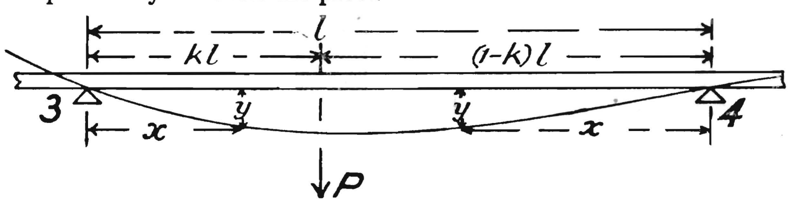


Fig. 3

Referring to the Figure

Let V = sum of all vertical forces to left of P (+ ve upward).

M_3 = B. Mt. at 3.

M_4 = B. Mt. at 4.

h_3, h_4 = heights of supports above datum.

P_2, P_3 = Loads of Spans l_2. l_3.

*See "Engineering Construction," by W. H. Warren, Chap. XI., &c.

† Proceedings S.U.E.S., 1903.

Then WITH ORIGIN AT 3. Curve to left of P.

B. Mt. 3 to P. $M = M_3 + Vx$ ... . (1)

$$EI\frac{dy}{dx} = M_3 x + \frac{V\ x^2}{2} + C_1 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$EI\ y = M_3\frac{x^2}{2} + V\frac{x^3}{6} + C_1 x + C_2 \ldots \quad \ldots \quad \ldots \quad (3)$$

C_2 is zero since when $x = o$, $y = o$.

$$\text{again } M_4 = M_3 + Vl - P(1 - k)\,l \quad \ldots \quad \ldots \quad (4)$$

$$\therefore V = \frac{M_4 - M_3}{l} + P(1 - k) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Substituting in (2) and (3) the value of V in (5).

$$EI\frac{dy}{dx} = M_3 x + \frac{M_4}{l}\frac{x^2}{2} - \frac{M_3}{l}\frac{x^2}{2} + P(1 - k)\frac{x^2}{2} + C_1 \quad (6)$$

$$EI\ y = M_3\frac{x^2}{2} + \frac{M_4}{l}\frac{x^3}{6} - \frac{M_3}{l}\frac{x^3}{6} + P(1 - k)\frac{x^3}{6} + C_1 x \quad (7)$$

Now for the curve to right of P by symmetry with origin at 4. Equations (6) and (7) are to be modified by putting k for $(1 - k)$ and C_3 for C_1 (C_3 is got from C_1 by putting $(1 - k)$ for (k) in C_1).

This may be realised clearly by putting the paper up towards the light and looking through the back of the paper: it is discussed in detail under the deflection of a beam, so that:

With origin at 4. Curve to right of P.

$$EI\frac{dy}{dx} = M_4 x + \frac{M_3}{l}\frac{x^2}{2} - \frac{M_4}{l}\frac{x^2}{2} + P(k)\frac{x^2}{2} + C_3 \quad (8)$$

$$EI\ y = M_4\frac{x^2}{2} + \frac{M_3}{l}\frac{x^3}{6} - \frac{M_4}{l}\frac{x^3}{6} + Pk\frac{x^3}{6} + C_3\,x \quad (9)$$

Note that there is no C_4 analogously with (3) and (7) above, this means a great saving of work in deducing the final result, as there are only two constants to be deduced.

At the point P the curve is continuous so that putting $x = kl$ in (6) and (7) and $x = (1 - k)\,l$ in (8) and (9) the $\frac{dy}{dx}$ is same in each, but of different sign and the y is the same in each with same sign; this follows from the fact that as x increases in (6) it decreases in (8), but the y increases or decreases in (8), as it increases or decreases in (9); the curves are of different equations as shewn in first part of this paper, but are tangential at the common point.

If the supports are on different levels, with differences of height say $h_n - h_{n-1}$, then y of $(7) = y \pm h_n - h_{n-1}$ of (9). The effects of supports being on different levels will be shewn by the portion in italics in the equations.

From equations (6) and (8), putting $x = kl$ right side of the equation (6) becomes.

$$M_3\,kl + \frac{M_4}{l}\frac{k\,l^2}{2} - \frac{M_3}{l}\frac{k^2l^2}{2} + P(1 - k)\frac{k^2l^2}{2} + C_1$$

$$\therefore \frac{dy}{dx} EI = \frac{M_3 \, lk \, (2 - k)}{2} + \frac{M_4 \, lk^2}{2} + \frac{Pl^2 \, (k^2) \, (1 - k)}{2} + C_1 \quad (10)$$

by analogy in equation (8).

$$-\frac{dy}{dx} EI = \frac{M_3 l \, (1 - k)^2}{2} + \frac{M_4 \, l \, (1 - k) \, (1 + k)}{2}$$

$$+ \frac{Pl^2 \, k \, (1 - k)^2}{2} + C_3 \quad \ldots \quad (11)$$

Since $\frac{dy}{dx}$ of (10) $= - \frac{dy}{dx}$ of (11).

$$\therefore - \left\{ C_1 + C_3 \right\} = \frac{M_3 l}{2} \left\{ (1 - k)^2 + k \, (2 - k) \right\} +$$

$$\frac{M_4 \, l}{2} \left\{ (1 - k^2) + k^2 \right\} + \frac{Pl^2}{2} \, k \, (1 - k) \left\{ \overline{1 - k} + k \right\}$$

$$= \frac{M_3 \, l + M_4 \, l + Pl^2 \, k \, (1 - k)}{2} \quad \ldots \quad \ldots \quad (12)$$

From equations (7) and (9), putting $x = kl$.

Equation (7) becomes

$$EI \, y = M_3 \frac{k^2 \, l^2}{2} + \frac{M_4}{l} \frac{k^3 \, l^3}{6} - \frac{M_3}{l} \frac{k^3 \, l^3}{6} + P \, (1 - k) \frac{k^3 \, l^3}{6} + C_1 \, kl,$$

$$i.e., \; EI \, y = \frac{M_3 \, l^2}{6} \left\{ 3k^2 - k^3 \right\} + \frac{M_4 \, l^2}{6} \, k^3 +$$

$$\frac{P \, l^3}{6} \, (1 - k) \, k^3 + C_1 \, kl \quad \ldots \quad \ldots \quad (13)$$

by analogy in equation (9).

$$EI \, y = \frac{M_3 \, l^2}{6} \left\{ (1 - k)^3 \right\} + \frac{M_4 \, l^2}{6} \left\{ 3 \, (1 - k)^2 - (1 - k)^3 \right\}$$

$$+ \frac{P \, l^3}{6} \, k \, (1 - k)^3 + C_3 \, (1 - k) \, l \quad \ldots \quad (14)$$

left side becomes EI $(y + h_4 - h_3)$ when supports are at different levels.

Equating the right sides of (13) and (14).

$$C_3 \, l - (C_1 + C_3) \, kl = \frac{M_3 \, l^2}{6} \left[\left\{ 3 \, k^2 - k^3 \right\} - (1 - k)^3 \right] +$$

$$\frac{M^4 \, l^2}{6} \left[k^3 - \left\{ 3 (1 - k)^2 - (1 - k)^3 \right\} \right] + \frac{Pl^3}{6} \left[(1 - k) \, k^3 - k \, (1 - k)^3 \right]$$

$$+ (h_4 - h_3) \, EI.$$

$$= \frac{M_3 \, l^2}{6} (3k - 1) + \frac{M^4 \, l^2}{6} \left[3k - 2 \right] + \frac{Pl^3}{6} (1 - k) \, (k) \, (2k - 1)$$

$$+ (h_4 - h_3) \, EI \quad \ldots \quad \ldots \quad (15)$$

From (12) and (15) substituting for $(C_1 + C_3)$

$$C_3 + \frac{3M_4 \, l + 3M_3 \, l + 3Pl^2 \, (k) \, (1 - k)}{6} \, k$$

$$= \frac{M_3 \, l \, (3k - 1) + M_4 \, l \, (3k - 2) + Pl^2 \, (1 - k) \, k \, (2k - 1)}{6} + \left(\frac{h_4 - h_3}{l} \right) EI$$

$$\therefore\ C_3 = -\frac{M_3\, l - 2M_4\, l - Pl^2\,(k)\,(1-k)\,(1+k)}{6} + \frac{h_4 - h_3}{l} EI.$$

$$= -\frac{M_3\, l + 2M_4\, l + Pl^2\, K_1}{6} + \frac{h_4 - h_3}{l}\ EI \quad \ldots \quad \ldots \quad (16)$$

by analogy or substituting in (12).

$$C_1 = -\frac{M_4\, l + 2M_3\, l + Pl^2\, K_2}{6} + \frac{h_3 - h_2}{l}\ EI \quad \ldots \quad \ldots \quad (17)$$

The investigation above is perfectly general.

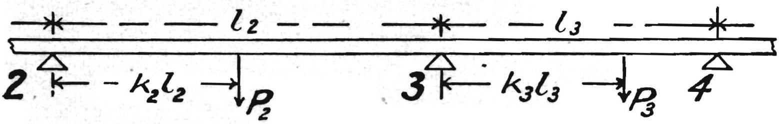


Fig. 4

and looking at Span 3-4. Fig. 4 slope at 3.

Origin at 3, x measured to right, here x = o.

$$\textit{i.e.},\ EI\frac{dy}{dx} = C_2 = -\frac{2M_3\, l_3 + M_4\, l_3 + P_3\, l_3\, {}^2K_2}{6} + \frac{h_4 - h_3}{l_3}\ EI \quad (18)$$

looking at Span 2-3. Slope at 3.

Origin at 3, x measures to left, here x = o.

$$EI\frac{dy}{dx} = C_2 = -\frac{M_2\, l_2 + 2M_3\, l_2 + P_2\, l_2\, {}^2K_1}{6} + \frac{h_3 - h_2}{l_2}\ EI \quad (19)$$

The $\frac{dy}{dx}$ of (18) $= -\frac{dy}{dx}$ of (19).

$$\therefore\ -\frac{2M_3\, l_3 + M_4\, l_3 + P_3\, l_3\, {}^2K_2}{6} + \frac{h_4 - h_3}{l_3}\ EI$$

$$= \frac{M_2\, l_2 + 2M_3\, l_2 + P_2\, l_2\, {}^2K_1}{6} - \frac{h_3 - h_2}{l_2}\ EI.$$

Putting Σ for sum.

$$\therefore\ M_2\, l_2 + 2M_3\,(l_2 + l_3) + M_4\, l_3 = -\Sigma\, P_2\, l_2\, {}^2K_1 - \Sigma\, P_3\, l_3\, {}^2K_2$$

$$\left(-\frac{h_3 - h_4}{l_3} - \frac{h_2 - h_3}{l_2}\right) 6\ EI \quad \ldots \quad \ldots \quad (20)$$

which is the Equation of Three Moments.

By putting in the amount of relative movements of supports (if any), the effects of end movements are readily deduced. For Distributed Loads (the equation is often quoted in this form).

Let w be the load per unit of length.

∴ Right hand side of the equation becomes—

$$-\int_0^{l_2} w_2\,(k - k^3)\, l_2^2\, dk\, l - \int_0^1 w_3 \left\{(1-k) - (1-k)^3\right\} l_3^2\, dk\, l.$$

$$= -\tfrac{1}{4}\, w_2\, l_2^3 - \tfrac{1}{4}\, w_3\, l_3^3$$

Thus Clapeyron's Equation is—

$$M_2\, l_2 + 2M_3\,(l_2 + l_3) + M_4\, l_3 = -\tfrac{1}{4}\, w_2\, l_2^3 - \tfrac{1}{4}\, w_3\, l_3^3 - \left\{\frac{h_3 - h_4}{l_3} - \frac{h_2 - h_3}{l^2}\right\} 6\, EI.$$

APPENDIX.

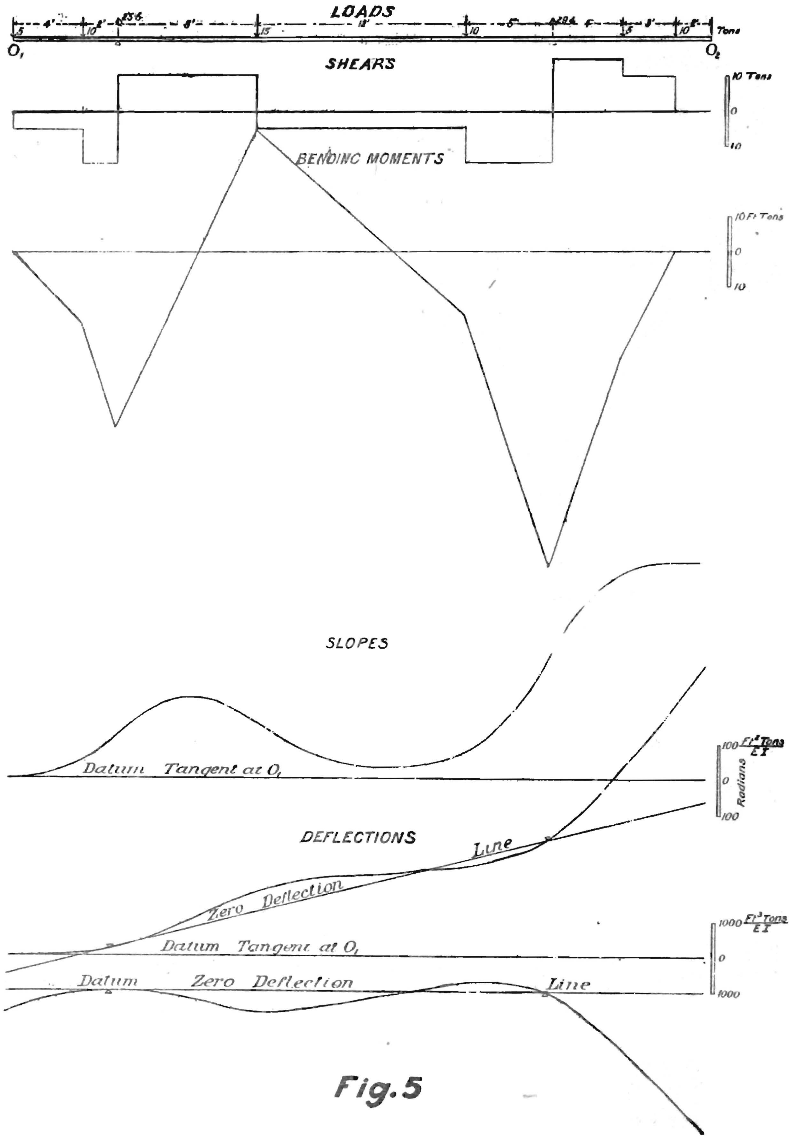


Fig. 5

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